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AN INFINITE FAMILY IN THE COHOMOLOGY OF THE STEENROD ALGEBRA

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1. Introduction

Let A denote the mod 2 Steenrod algebra, $\mathrm{Ext}_A(\mathbb{Z}_2)$ its cohomology, and h_i the nonzero element of $\mathrm{Ext}_A^{1,2^i}(\mathbb{Z}_2)$. Inspection of Tangora's tables [6] leads one to conjecture that h_i is acted upon faithfully by $\mathrm{Ext}_A(\mathbb{Z}_2)$ until the relation $h_{i-1}h_i = 0$.

Conjecture 1.1. *If $\alpha \neq 0 \in \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2)$ for $0 < t-s < 2^{i-1}-1$, then $\alpha h_i \neq 0$.*

Our main result is that the qualitative content of this conjecture is true, although our results are not this sharp. We show that the h_i are acted on faithfully by portions of $\mathrm{Ext}_A(\mathbb{Z}_2)$ which increase with i. Alternatively, every nonzero element of $\mathrm{Ext}_A(\mathbb{Z}_2)$ is acted upon nontrivially by h_i for i sufficiently large.

A simply stated, albeit weakened, form of our main result is the following.

Theorem 1.2. *If $\alpha \neq 0 \in \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2)$ for $4 \leqslant t-s \leqslant 2^j$, then $\alpha h_i \neq 0$ for all $i \geqslant 2j+1$.*

Thus we require the subscript in h_i to be roughly twice as large as the conjectured minimal value.

In Section 3 we consider the subalgebra H of $\mathrm{Ext}_A(\mathbb{Z}_2)$ generated by the h_i's. We conjecture that all relations in H are obtained from the Steenrod operations Sq^I acting on the relations h_0h_1 and $h_0h_2^2$, and give what we conjecture to be a complete set of relations. We give a nice inductive description of H if this conjecture is true and use the results and methods of Section 2 to show that many of these conjectured elements are nonzero.

The author wishes to acknowledge significant contributions to this work by Mark Mahowald and Frank Adams. Mahowald mentioned Conjecture 1.1 to the author in 1976. A (successful) attempt to prove Theorem 1.2 led to the contribution of Mahowald and the author to the work of W.H. Lin [4]. Our interest in Theorem 1.2

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was stimulated by our work on v_i-periodicity [3; 3.10(i)]. If $H^s(\ )$ denotes the stable Hopf invariant in $\mathrm{Ext}_A(\mathbb{Z}_2)$ and $\mathscr{F}$ denotes a known family of elements in $\mathrm{Ext}_A(\mathbb{Z}_2)$, one might define a new family as $\{H^s(\alpha h_i): i \text{ large}, \alpha \in \mathscr{F}\}$. Theorem 1.2 provides an essential ingredient to this study—that αh_i be nonzero for i large. The proof of Theorem 1.2 presented here is based upon one given by Adams.

2. Faithful action on h_i

Let A_r denote the subalgebra of A generated by $\mathrm{Sq}^1, \ldots, \mathrm{Sq}^{2^r}$. A stronger version of Theorem 1.2 is the following.

Theorem 2.1. *If* $\alpha \in \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2)$ *has nonzero reduction to* $\mathrm{Ext}_{A_r}^{s,t}(\mathbb{Z}_2)$ *and* $2^i > (2^{r+2}-1)s - t + 1$, *then* $\alpha h_i \neq 0$.

Proof. As in [4], let $P = \mathbb{Z}_2[x, x^{-1}]$ with $\mathrm{Sq}^i x^j = \binom{j}{i} x^{j+i}$ for any integer j. There is an isomorphism

$$\mathrm{Ext}_{A_{r+1}}^{*,*}(\Sigma P) \approx \bigoplus_{j \in \mathbb{Z}} \mathrm{Ext}_{A_r}^{*,*}(\Sigma^{2^{r+2}j}\mathbb{Z}_2).$$

This was the key idea of [4], but was not stated there as a numbered result. It follows from Lemmas 1.3 and 1.6 of that paper, and is dual to an isomorphism stated midway through the proof of [4; 1.1]. This isomorphism implies that $\alpha \cdot \Sigma 1_{2^i-1}$ is a nonzero element of $\mathrm{Ext}_{A_{r+1}}^{s,t+2^i}(\Sigma P)$.

Let $P_1 = \tilde{H}^*(RP^\infty)$, which is the submodule of P consisting of terms of positive degree. If $P^0 = P/P_1$, then $\mathrm{Ext}_{A_{r+1}}(\Sigma P^0)$ may be calculated from a spectral sequence beginning with $\bigoplus_{j \leq 1} \mathrm{Ext}_{A_{r+1}}(\Sigma^j \mathbb{Z}_2)$. The May spectral sequence (see [6]) implies that $\mathrm{Ext}_{A_{r+1}}^{p,q}(\mathbb{Z}_2) = 0$ if $q > (2^{r+2}-1)p$, so that $\mathrm{Ext}_{A_{r+1}}^{p,q}(\Sigma P^0) = 0$ if $q > (2^{r+2}-1)p + 1$. Since $t + 2^i > (2^{r+2}-1)s + 1$, the exact sequence in $\mathrm{Ext}_{A_{r+1}}(\ )$ induced by $0 \to P_1 \to P \to P^0 \to 0$ implies that $\alpha \cdot \Sigma 1_{2^i-1}$ is nonzero in $\mathrm{Ext}_{A_{r+1}}^{s,t+2^i}(\Sigma P_1)$.

The Adem relations easily imply that the function $\Sigma P_1 \xrightarrow{\varphi} \overline{A//A_r}$ defined by $\varphi(\Sigma x^i) = [\mathrm{Sq}^{i+1}]$ is a homomorphism of A_{r+1}-modules. Here $\overline{A//A_r}$ denotes the elements of positive degree and $[\mathrm{Sq}^{i+1}]$ denotes the equivalence class (possible 0) in the quotient module. $\varphi^*: \mathrm{Ext}_{A_{r+1}}(\overline{A//A_r}) \to \mathrm{Ext}_{A_{r+1}}(\Sigma P_1)$ sends $1_{\mathrm{Sq}(2^i)} \mapsto \Sigma 1_{2^i-1}$. Thus $\alpha \cdot 1_{\mathrm{Sq}(2^i)}$ is nonzero in $\mathrm{Ext}_{A_{r+1}}(\overline{A//A_r})$ and hence also in $\mathrm{Ext}_A(\overline{A//A_r})$. The theorem now follows from the exact sequence

$$\mathrm{Ext}_A^{s,t+2^i}(A//A_r) \to \mathrm{Ext}_A^{s,t+2^i}(\overline{A//A_r}) \xrightarrow{\delta} \mathrm{Ext}_A^{s+1,t+2^i}(\mathbb{Z}_2),$$

since $\delta(1_{\mathrm{Sq}(2^i)}) = h_i$ and the first group is $\mathrm{Ext}_{A_r}^{s,t+2^i}(\mathbb{Z}_2) = 0$ by the vanishing line argument of the previous paragraph.

Since $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2) \to \mathrm{Ext}_{A_r}^{s,t}(\mathbb{Z}_2)$ is an isomorphism for $t - s < 2^{r+1} - 1$, we may immediately deduce the following.

Corollary 2.2. *Let $p(m)$ denote the smallest 2-power $> m$. If $\alpha \neq 0 \in \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2)$, then $\alpha h_i \neq 0$ if $2^i > (2p(t-s+1)-1)s-t+1$.*

Theorem 1.2 follows from Corollary 2.2 once we observe that $s \leq 2^{i-1}$ if $4 \leq t-s \leq 2^i$.

We give in Table 2.3 for $t-s \leq 13$ the results implied by our theorems.

Table 2.3

	$\alpha h_i \neq 0$ for $i \geq$					
α	s	t	1.2	2.2	2.1	actual
h_1	1	2	*	3	3	3
h_1^2	2	4	*	4	4	3
h_2	1	4	5	4	4	4
h_0h_2	2	5	5	5	5	4
$h_0^2h_2$	3	6	*	6	6	4
h_2^2	2	8	7	5	5	4
c_0	3	11	7	7	6	4
h_1c_0	4	13	9	7	6	4
Ph_1	5	14	9	8	6	5
Ph_1^2	6	16	9	8	7	5
Ph_2	5	16	9	8	6	5
Ph_0h_2	6	17	9	8	7	5
$Ph_0^2h_2$	7	18	9	8	7	5
h_3	1	8	7	5	5	5
h_0h_3	2	9	7	6	6	5
$h_0^2h_3$	3	10	7	7	7	5
$h_0^3h_3$	4	11	7	7	7	5
h_1h_3	2	10	7	6	6	5
$h_1^2h_3$	3	12	9	7	7	5

Our estimates for all elements with $s=2$ may be improved to best possible by using the stronger vanishing line $\mathrm{Ext}_{A_r}^{2,t}(\mathbb{Z}_2)=0$ if $t>2^{r+1}$. The significance of our results is that they hold for elements far beyond the range in which calculations have been made.

Inspection of the charts of [6] also leads one to state the following.

Conjecture 2.4. *If $\alpha \neq 0 \in \mathrm{Ext}_A^{s,t}(\mathbb{Z}_2)$ for $0<t-s<2^{i-2}-2$, then $\alpha h_i^2 \neq 0$.*

This conjecture says that the first relation on h_i^2 is $h_{i-3}^2h_i^2$. We have not been able to prove even a qualitative version of this conjecture.

3. The subalgebra H

Proposition 3.1. *The following elements are 0 in $\mathrm{Ext}_A(\mathbb{Z}_2)$. They are all obtainable from h_0h_1 and $h_0h_2^2$ by using the squaring operations Sq^I.*

$$h_0^{2^i}h_{i+1}\quad (i\geqslant 2);\qquad h_0^{2^i}h_{i+2}^2,\quad h_0^{2^i}h_{i+3}^3,\quad h_ih_{i+1},\quad h_ih_{i+2}^2,\quad h_ih_{i+3}^3,$$
$$h_i^2h_{i+2}+h_{i+1}^3,\quad h_i^2h_{i+3}^2,\quad h_i^2h_{i+4}^3\quad (i\geqslant 0).$$

Remark. The relation h_{i+1}^4 is derivable from these as $h_{i+1}(h_i^2h_{i+2}+h_{i+1}^3)$.

Proof. Liulevicius [5] was perhaps the first to note that there are squaring operations $\mathrm{Sq}^i:\mathrm{Ext}_A^{k,j}(\mathbb{Z}_2)\to\mathrm{Ext}_A^{k+i,2j}(\mathbb{Z}_2)$ which satisfy

(i) $\mathrm{Sq}^i\alpha=0$ if $\alpha\in\mathrm{Ext}^{k,j}$ with $k<i$,
(ii) $\mathrm{Sq}^k\alpha=\alpha^2$ if $\alpha\in\mathrm{Ext}^{k,j}$,
(iii) $\mathrm{Sq}^i(\alpha\beta)=\sum\mathrm{Sq}^j\alpha\cdot\mathrm{Sq}^{i-j}\beta$,
(iv) $\mathrm{Sq}^0h_j=h_{j+1}$.

The relations $h_0h_1=0$ and $h_0h_2^2=0$ were established in [1]. Applying $(\mathrm{Sq}^0)^i$ increases all subscripts by i. $h_0^2h_2+h_1^3$, $h_0^2h_3^2$, and $h_0h_3^3$ are obtained as $\mathrm{Sq}^1(h_0h_1)$, $\mathrm{Sq}^1(h_0h_2^2)$, and $h_0\cdot\mathrm{Sq}^0\mathrm{Sq}^0\mathrm{Sq}^1(h_0h_1)$. We also have

$$h_0^{2^i}h_{i+1}=\mathrm{Sq}^{2^{i-1}}\cdots\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1(h_0h_1)\quad\text{if } i\geqslant 2,$$
$$h_0^{2^i}h_{i+2}^2=\mathrm{Sq}^{2^{i-1}}\cdots\mathrm{Sq}^2\mathrm{Sq}^1(h_0h_2^2),$$
$$h_0^{2^i}h_{i+3}^3=\mathrm{Sq}^{2^{i-1}}\cdots\mathrm{Sq}^1(h_0h_3^3),$$

and $h_0^2h_4^3=\mathrm{Sq}^1(h_0h_3^3)$.

Conjecture 3.2. *The relations listed in Proposition* 3.1 *are a complete set of relations for the subalgebra H of $\mathrm{Ext}_A(\mathbb{Z}_2,\mathbb{Z}_2)$ generated by the h_i's; i.e. H is $\mathbb{Z}_2[h_0,h_1,\ldots]/I$ where I is the ideal generated by the elements listed in Proposition* 3.1.

Two nice inductive descriptions of this algebra $Q=\mathbb{Z}_2[h_0,h_1,\ldots]/I$ can be given.

Description 3.3. For $j\geqslant 0$ let U_j denote the elements of Q with $t-s\leqslant j$. Let $U_{-2}=U_{-1}=U_0=\{h_0^i: i\geqslant 0\}$. Then

$$Q=U_0\oplus\{h_1^2,h_1^3,h_2^3\}\oplus\bigoplus_{i\geqslant 0}\{h_{i+1},h_{i+2}^2,h_{i+3}^3\}\cdot U_{2^i-3}/h_0^{2^i}.$$

This description is only valid as a vector space. It fails to describe the relations $h_i^2h_{i+2}=h_{i+1}^3$. It incorporates nicely the way in which each initial segment of U occurs three times. See for example Fig. 3.1. Of course, these also occur as subsets of longer initial segments.

Description 3.4. Let $S_{-1}=S_0=T_{-1}=T_0=\{h_0^i\}$, $T_1=\{h_0^i,h_1\}$, $S_1=\{h_0^i,h_1,h_1^2\}$, and $T_2=\{h_0^i,h_1,h_1^2,h_2,h_0h_2,h_0^2h_2=h_1^3\}$. For $i\geqslant 2$, let

$$S_i=T_i\oplus(T_{i-3}/h_0^{2^{i-2}})\cdot h_i^2\qquad T_{i+1}=S_i\oplus(S_{i-1}/h_0^{2^i})\cdot h_{i+1}.$$

Then $Q=\varinjlim S_i=\varinjlim T_i$.

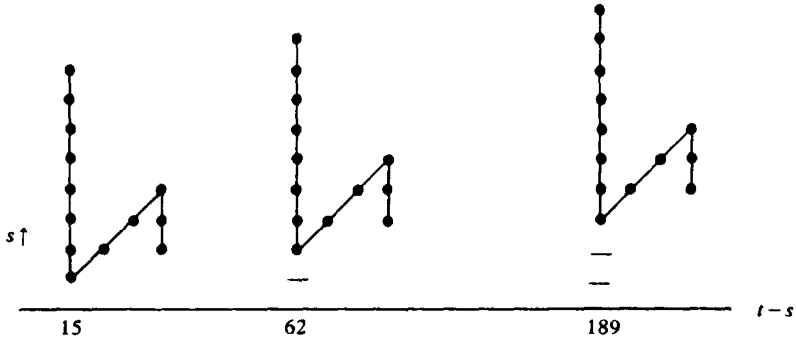

Fig. 3.1.

This description incorporates nicely the way in which Q is generated by $\{h_i, h_i^2\}$. That many of the elements of Q are nonzero in H is implied by Theorem 2.1 (and the strengthening when $s=2$ mentioned after Table 2.3). However some elements such as $h_0^2h_2h_4$, $h_0^2h_2h_5$, and $h_0^2h_3h_5$ are not easily handled by our methods. Also, the elements $h_0^jh_i^\varepsilon$ for $\varepsilon = 1, 2, 3$ are not handled by Theorem 2.1. However, a proof in the spirit of this paper for the elements $h_0^jh_i$ is now presented.

Proposition 3.5. *$h_0^{2^i-1}h_{i+1}$ is nonzero in* $\mathrm{Ext}_A(\mathbb{Z}_2)$.

Proof. Consider the homomorphisms

$$\mathrm{Ext}_A(\overline{A//A_0}) \xrightarrow{\rho} \mathrm{Ext}_{A_1}(\overline{A//A_0}) \xrightarrow{\varphi^*} \mathrm{Ext}_{A_1}(\Sigma P_1),$$

where φ is as in the proof of Theorem 2.1. The composite sends $h_0^j 1_{\mathrm{Sq}(2^{i+1})}$ to $h_0^j \cdot \Sigma 1_{2^{i+1}-1}$. But $\mathrm{Ext}_{A_1}(\Sigma P_1)$ is easily calculated to be as given in Fig. 3.2 (e.g. [2; 3.4(i)]). Thus $h_0^{2^i-1}\Sigma 1_{2^{i+1}-1} \neq 0$. The proposition now follows from the fact that $\mathrm{Ext}_A^{s,t}(\overline{A//A_0}) \xrightarrow{\delta} \mathrm{Ext}_A^{s+1,t}(\mathbb{Z}_2)$ is injective in $t-s>0$ since its kernel is $\mathrm{Ext}_A^{s,t}(A//A_0) \approx \mathrm{Ext}_{A_0}^{s,t}(\mathbb{Z}_2)$.

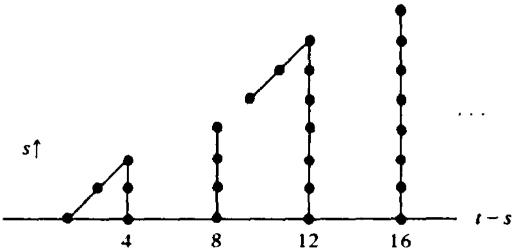

Fig. 3.2.

References

[1] J.F. Adams, On the non-existence of elements of Hopf invariant one, Ann. of Math. 72 (1960) 20–104.
[2] D.M. Davis and M. Mahowald, Obstruction theory and ko-theory, in: Lecture Notes in Math. 658 (Springer, Berlin-New York, 1978) 134–164.
[3] D.M. Davis and M. Mahowald, v_1- and v_2-periodicity in stable homotopy, Amer. J. Math., to appear.
[4] W.H. Lin, D.M. Davis, M. Mahowald and J.F. Adams, Calculation of Lin's Ext groups, to appear in Math. Proc. Camb. Phil. Soc. 87 (1980) 459–469.
[5] A. Liulevicius, The factorization of cyclic reduced powers by secondary cohomology operations, Mem. Amer. Math. Soc. 42 (1962).
[6] M. Tangora, On the cohomology of the Steenrod algebra, Math. Z. 116 (1970) 18–64.