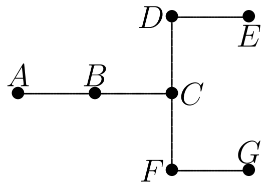
2022 LEHIGH UNIVERSITY HIGH SCHOOL MATH CONTEST

Number in [brackets] is the number of correct answers, out of 114.

1. [86] (2 pts) What is the sum of the coefficients of the polynomial $p(x) = (x^2 - x - 1)^{2022}$?
2. [94] (2 pts) If $a_1 = 2$, $a_2 = 3$, $a_3 = 5, \ldots$ is the sequence of all positive integers which are not perfect squares, what is a_{1000}?
3. [100] (2 pts) Let $f(x) = 1/(1 + 2x)$. What is the sum of all numbers x satisfying $f(f(x)) = x$?
4. [86] (2 pts) What is the largest prime number p such that $p^2 - 1$ has exactly 10 positive divisors (including 1 and $p^2 - 1$)?
5. [61] (2 pts) If S is the sum of all the real roots of the equation $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$, and P is their product, what is the value of $\lfloor S + P \rfloor$?
6. [82] (2 pts) Four circles of radius 1 are arranged symmetrically so that each is tangent to two others. What is the ratio (> 1) of the radii of the two circles which can be formed tangent to all four circles?
7. [37] (2 pts) What is the sum of all 2-digit numbers $x \geq 10$ for which the sum of the digits of kx equals the sum of the digits of x for all values of k from 2 to 10?
8. [70] (2 pts) The 12-digit integer 999999995904 can be factored as $a^s \cdot b^t \cdot c \cdot d \cdot e \cdot f$, where s and t are integers > 1 and $a < b < c < d < e < f$ are primes. Compute $s + f$.
9. [35] (3 pts) Let $X = (10/9)^{9/10}$, $a_1 = X$, and, for $n \geq 2$, $a_n = X^{a_{n-1}}$. Find the integer closest to $1000a_{1000}$.

10. [55] (3 pts) In the diagram below, the distance between adjacent vertices is 1. How many paths of length 5 starting at B are there? A path goes from any vertex to any adjacent vertex. For example, B to C to B is a legitimate path of length 2, as is B to C to D.



11. [55] (3 pts) In parallelogram $ABCD$, angle A is acute and $AB = 5$. Point E is on AD with $AE = 4$ and $BE = 3$. Point F lies on the line CD, with BF perpendicular to that line. If $BF = 5$, find EF.

12. [85] (3 pts) Kim and Chris live in a skyscraper with 10 apartments on each floor. Apartments 1-10 are on Floor 1, apartments 11-20 on Floor 2, etc. Kim's apartment is on the floor whose number coincides with the number of Chris's apartment. If the sum of the numbers of their apartments is 239, what is the number of Kim's apartment?

13. [48] (3 pts) Let $T(b, n)$ denote a tower of repeated exponentials of b with n copies of b. For example, $T(2, 4) = 2^{2^{2^2}} = 2^{16} = 65536$. What is the smallest positive integer n such that $T(2, n) \geq T(16, 4)$?

14. [63] (3 pts) A fourth-degree polynomial $p(x)$ has values as in the following table. What is $p(7)$?

x	1	2	3	4	5	6
$p(x)$	8	26	72	188	440	918

15. [52] (3 pts) Three married couples sit randomly around a circular table. What is the probability that no couples are in adjacent seats?

16. [56] (3 pts) Every cell of a 3×3 table contains an integer. The sum of the numbers in each column (except the first) equals 4 times the sum in the previous column. The sum of the numbers in each row (except the first) exceeds that in the previous row by 6. The sum of the entries in one of the rows is 2022. What is the sum of the numbers in the first column?

17. [37] (3 pts) What is the length of the shortest path in the plane which starts and ends at the point $(1, 7)$, and intersects the y-axis and the line $y = x$?

18. [39] (3 pts) What is the smallest positive multiple of 99 such that all of its digits are even?

19. [32] (4 pts) What is the area of the largest square that can be inscribed in a unit cube? Each vertex of the square must lie on an edge of the cube.

20. [42] (4 pts) What is the largest integer that divides both $n^2 + 1$ and $(n + 1)^2 + 1$ for some positive integer n?

21. [35] (4 pts) What reduced fraction p/q with $8 \leq q \leq 99$ is closest to $5/7$?

22. [21] (4 pts) You roll a fair 6-sided die ten times in succession. What is the expected number of times that you roll a 6 exactly three times in a row? (Four times in a row does not count.)

23. [42] (4 pts) In how many ways can a cube be colored in six distinct colors if rotated versions of a coloring are considered to be the same?

24. [4] (4 pts) In a cube with sides of length 2, a circle is inscribed in three mutually-adjacent square faces. A point is chosen on each of the circles. What is the maximum perimeter of the triangle with these three points as vertices?

25. [8] (5 pts) A certain type of organism does not interact with others, but can spontaneously reproduce. If left alone for an hour, the result will be, with equal probabilities, 0, 1, 2, or 3 organisms, reflecting various possibilities of dying or reproducing. Resulting organisms will behave in the same way during subsequent hours. If you start with one organism, let p be the probability that the population is self-sustaining, i.e., never dies off. What integer is closest to $100p$?

26. [16](5 pts)An ant moves in the plane 300 length-1 steps. It turns 90 degrees left 99 times, and 90 degrees right 200 times. What is the maximum distance between its start and end points?

27. [32] (5 pts) Let ABC be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Let D and E be the feet of the altitudes from A and B, respectively. What is the diameter of the circumcircle of triangle CDE?

28. [3] (5 pts) Consider the set S of all ordered pairs (a, b) of integers between 1 and 12 inclusive such that the mod-13 value of $a^n + b^{n+9}$ is the same for all positive integers n. What is the sum of the first coordinates of the pairs in S?

29. [13] (5 pts) Let b be a positive real number, and let a_n be a sequence defined by $a_1 = a_2 = a_3 = 1$, and $a_n = a_{n-1} + a_{n-2} + ba_{n-3}$ for $n \geq 4$. What is the smallest b such that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{4^n}$ diverges?

30. [0] (5 pts) Sixty-four students are ranked from 1 (worst) to 64 (best) in terms of math ability, with no ties. There are eight clubs, numbered 1 to 8, with club 8 also considered as club 0, with eight students in each club. For i from 1 to 8, there is a contest between club i and club $i - 1$, in which each student in club i competes against each student in club $i - 1$, and the higher-ranked student always wins. It turns out that club i wins at least K matches against club $i - 1$ for all values of i (1 to 8). What is the largest value of K for which this can happen?

SOLUTIONS.

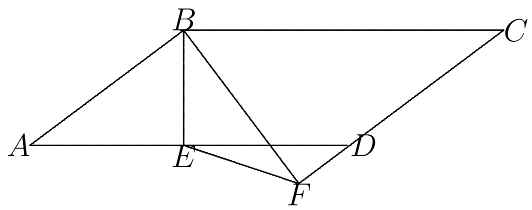
1. 1. It is $p(1) = (-1)^{2022} = 1$.
2. 1032. Since $32^2 = 1024$, $a_{993} = 1025$. Thus $a_{1000} = 1032$.
3. $-1/2$. $f(f(x)) = \frac{1+2x}{3+2x}$. Setting this equal to x yields the equation $2x^2 + x - 1 = 0$ with solutions -1 and $1/2$. In fact, these satisfy $f(x) = x$.
4. 7. Clearly $p \neq 2$ or 3. Then $p^2 - 1$ is divisible by 2 and 3. The only numbers which are divisible by 2 and 3 and have exactly 10 divisors are $2^4 \cdot 3 = 48 = 7^2 - 1$ and $3^4 \cdot 2 = 162$. Thus $p = 7$ is the only number that works.
5. 4. Add x^2 to both sides, obtaining $(x-1)^4 = x^2$, so $(x-1)^2 = \pm x$. Then $(x-1)^2 = -x$ has no solutions, but $(x-1)^2 = x$ has solutions $x = \frac{1}{2}(3 \pm \sqrt{5})$. Thus $S = 3$ and $P = 1$.
6. $3+2\sqrt{2}$. Let r be the radius of the little circle. A right triangle implies that $(r+1)^2 + (r+1)^2 = 2^2$, so $r = \sqrt{2} - 1$, and then the larger radius equals $\sqrt{2} + 1$. The ratio equals $3 + 2\sqrt{2}$.
7. 252. The sum of the digits of $9x$ is a multiple of 9, and hence so also is the sum of the digits of x. The answer is that 18, 45, 90, and 99 work. 27×7, 63×3, 36×8, 72×4, and 81×9 show that those do not work. One easily checks, by successive addition, that the others do work.
8. 617. The number equals $10^{12} - 2^{12} = 2^{12}(5^{12} - 1)$. Since
$$5^{12} - 1 = (5^3 - 1)(5^3 + 1)(5^2 + 1)(5^4 - 5^2 + 1),$$
we obtain that the desired factorization is $2^{16} \cdot 3^2 7 \cdot 13 \cdot 31 \cdot 601$.

9. 1111. Let Y denote the limiting value of the a_n sequence. Then $Y = X^Y$, or $X = Y^{1/Y}$. Since $X = (10/9)^{1/\frac{10}{9}}$, we deduce that $Y = 10/9$.

10. 32. In general, there are 2^n paths of length n. This is proved by induction. Assume the result known for $n = 2k$. After $2k$ steps, you will be at B, D, or F, and there are 2 possible steps from each. So the result is true for $2k + 1$. Also, from B there are 4 possible paths of length 2, and similarly from D or F. Therefore, the result is true for $2k + 2$.

11. $\sqrt{10}$. Since BF is also perpendicular to AB, $\cos(EBF) = \sin(ABE) = 4/5$. Thus

$$EF^2 = BE^2 + BF^2 - 2 \cdot BE \cdot BF \cdot \tfrac{4}{5} = 9 + 25 - 24.$$



12. 217. If Kim's apartment is on floor C and has number $10(C - 1) + x$ with $1 \le x \le 10$, then $10(C - 1) + x + C = 239$, so $11C = 249 - x$. The only number from 239 to 248 divisible by 11 is 242, so $x = 7$ and $C = 22$. Thus $10(C - 1) + x = 217$.

13. 7. $\log_2(\log_2(T(16, 4)) = \log_2(4 \cdot T(16, 3)) = 2 + 4 \cdot 16^{16} = 2 + 2^{66}$. Certainly

$$T(2, 4) = 2^{16} < 2 + 2^{66} < 2^{2^{16}} = T(2, 5).$$

Thus $T(2, 6) < T(16, 4) < T(2, 7)$.

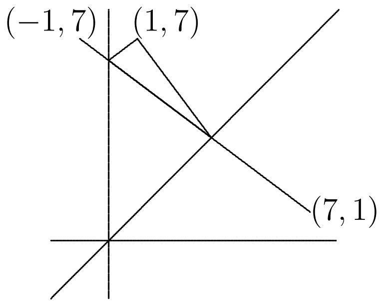
14. 1736. If you do four rounds of successive differences, you get to a constant sequence 24, 24. Extending this one place farther and backing up leads to the answer.

15. 4/15. Mr A chooses a seat. There are 120 choices for where the other five sit. Suppose Mrs A sits directly across from him. Then there are 4 choices for where Mr B sits, 2 choices for who sits next to him, and 2 choices for the order of the other two, so 16 possibilities. Alternatively, there are 2 other choices for

Mrs A, then 4 choices for who sits between Mr A and Mrs A. The spouse of that person must be in the middle of the other three, so there are 2 choices for the order of those three, yielding another 16 possibilities. The answer is 32/120.

16. 288. If S is the sum in the first column, then the sum of all numbers in the table is $S+4S+16S=21S$, so is divisible by 7. The row sums must be three consecutive of 2010, 2016, 2022, 2028, and 2034, with mod-7 values of $1,0,-1,-2$, and -3. To have this be 0, we must choose the first three. So the total sum is $3\cdot 2016$, and $S=2016/7=288$.

17. 10. The path should be a triangle with one vertex at $(1,7)$ and one on each of the specified lines. Reflecting across the two lines shows that the shortest path will have the same length as the line from $(-1,7)$ to $(7,1)$, and this length equals 10.



18. 228888. The digit sum must be a multiple of 18, and the alternating digit sum a multiple of 22, and their difference a multiple of 4. Make the sum in even positions 18, and in odd positions 18, and quickly get this answer.

19. 9/8. If the vertices of the cube are at points whose coordinates are 0 or 1, then, by symmetry, the vertices of the square should be at $(x,0,0)$, $(1,0,1-x)$, $(1-x,1,1)$, and $(0,1,x)$ for some number x. Equating two of the squared lengths yields $2x^2+1=2(1-x)^2$, so $x=\frac{1}{4}$, and the the squared length equals 9/8.

20. 5. If d is such an integer, then d divides $(n+1)^2+1-(n^2+1)=2n+1$. Hence d also divides $(2n+1)^2-4(n^2+1)=4n-3$.

Finally d divides $2(2n+1)-(4n-3)=5$. Using $n=2$, we see that $d=5$ works.

21. 68/95. Since $\left|\frac{p}{q}-\frac{5}{7}\right|=\frac{|7p-5q|}{7q}$, we look for the largest $q \leq 99$ for which there exists an integer p with $7p-5q=\pm 1$. So we want a multiple of 7 slightly less than 500 which is $\equiv \pm 1 \bmod 5$. We find that $p=68$ works, with $68 \cdot 7 = 5 \cdot 95 + 1$, so $\frac{68}{95}-\frac{5}{7}=\frac{1}{7 \cdot 95}$ clearly does the best job.

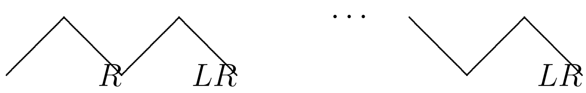
22. 35/1296. You just add together the probabilities of each of the ways that it can happen. There are two ways with three 6s at an end, next to a non-6. That gives $2(\frac{1}{6})^3(\frac{5}{6})$, and there are six ways of having three 6s with a non-6 on either side. This gives $6(\frac{1}{6})^3(\frac{5}{6})^2$. The total is $(\frac{1}{6})^3(\frac{35}{6})=\frac{35}{1296}$.

23. 30. For a fixed position of the cube, there are 6! colorings. Each color combination has $6 \cdot 4$ orientations: 6 directions for Color 1 to point in, and then 4 ways to rotate it around the axis perpendicular to the face with Color 1. Therefore the answer is $6!/24$.

24. $3\sqrt{6}$. We place the center of the cube at $(0,0,0)$ and let the three faces be the intersection of the cube with the planes $x=1$, $y=1$, and $z=1$, respectively. The points $(1,0,-1)$, $(-1,1,0)$, and $(0,-1,1)$ will give this maximum perimeter. To show that this is the maximum, you can let the points be $(1,\cos\alpha_1,\sin\alpha_1)$, $(\sin\alpha_2,1,\cos\alpha_2)$, and $(\cos\alpha_3,\sin\alpha_3,1)$. You can show that the sum of the squares of the lengths is $18-2\sum(1+\sin\alpha_i)(1+\cos\alpha_{i+1})$. This has a maximum value of 18, and by the Cauchy-Schwarz Inequality, the sum of the lengths is $\leq \sqrt{3 \cdot 18}$.

25. 59. We will show that the value of p is $2-\sqrt{2} \approx .586$. If q denotes the probability that the population will eventually die off, then q^n is the probability that a population of n organisms will eventually die off. Thus $q=\frac{1}{4}(1+q+q^2+q^3)$. This is equivalent to the equation $(q-1)(q^2+2q-1)=0$. Only the solution $q=\sqrt{2}-1$ makes sense, and $p=1-q$.

26. $100\sqrt{2}$. The path $R(LR)^{99}R^{100}$ indicated below has length $100\sqrt{2}$. The final R^{100} makes 25 cycles and is not shown. A path which starts with L can be decomposed as $L(R^{k_1}L)\cdots(R^{k_{98}}L)R^{k_{99}}$ with each completed step of length $\leq \sqrt{2}$, and total of length $\leq 100\sqrt{2}$ by the triangle inequality. Similarly $R^{k_1}(LR^{k_2})\cdots(LR^{k_{100}})$ shows that a path starting with an R has length $\leq 100\sqrt{2}$.



27. 39/4. Let AD and BE intersect at X. Since CDX and CEX are right angles, $CDXE$ is a cyclic quadrilateral, and CX is the diameter of the circumcircle of CDE. We obtain

$$\begin{aligned} 169 &= AB^2 = BC^2 + AC^2 - 2(BC)(AC)\cos(C) \\ &= 196 + 225 - 2(14)(15)\cos(C), \end{aligned}$$

so $\cos(C) = 3/5$. Thus triangles CDA, CEB, and XDB are all similar to 3-4-5 triangles. We obtain $CD = 9$, $BD = 5$, and $DX = \frac{15}{4}$. Finally $CX = 39/4$ by Pythagoras.

28. 27. The answer is that (a, b) can be (1,1), (4,4), (10,10), or (12,12). In this solution, $\equiv$ always means mod 13. By Fermat's Little Theorem, $a^{12} \equiv b^{12} \equiv 1$. Letting $n = 3$ and 12, we obtain $a^3 + 1 \equiv 1 + b^9$, so $a^3 \equiv b^9$. For all n, $a^n + a^3b^n \equiv a^n + b^{n+9} \equiv a^{n+3} + b^n$, so $(a^n - b^n)(1 - a^3)$. (Case 1): $a^3 \equiv 1$. Then $b^3 \equiv a^3b^3 \equiv b^{12} \equiv 1$. Thus $a^n + b^n \equiv a^n + b^{n+9}$ is the same for all n. Therefore, $a + b \equiv a^3 + b^3 \equiv 2$, and so $a = b = 1$, since the only numbers whose cubes are $\equiv 1$ are 1, 3, and 9. (Case 2): If $a^3 \not\equiv 1$, then $a^n \equiv b^n$ for all n. Hence $a^9 \equiv b^9 \equiv a^3$. Thus $a^3(a^3 - 1)(a^3 + 1) \equiv 0$, so $a^3 \equiv -1$, which implies $a = 4$, 10, or 12.

29. 44. Let r be the positive root of the polynomial $P(x) = x^3 - x^2 - x - b$. By induction, one can show that there exist positive constants c_1 and c_2 such that $c_1 r^n \le a_n \le c_2 r^n$ for all positive integers n. If $r < 4$, $\sum a_n/4^n \le c_2 \sum (r/4)^n$, so the series converges, and similarly if $r \ge 4$, the series diverges. If $b = 44$, then $P(4) = 0$, so the series diverges, while if $b < 44$, $P(4) > 0$ so $r < 4$ and the series converges.

30. 44. Suppose the students are placed in clubs as in the following table.

1	2	3	4	5	6	7	8
1	2-3	4-6	7-10	11-15	16-21	22-28	29-36
37-43	44-49	50-54	55-58	59-61	62-63	64	

Club i will win $8(8 - i) + i(i - 1)$ matches against club $i - 1$. The minimum value of this for integers i occurs when $i = 4$ or 5, and is 44.

To see that K must be ≤ 44, consider any distribution, and choose the 4th best student in each club. The club with the weakest of these students will lose at least $4 \cdot 5$ matches competing against any other club.