2014 Lehigh contest, annotated by the number of people answering correctly out of the 49 people who got at least 22 right.

1. [48] Simplify $\frac{1}{2}/(\frac{1}{3}-\frac{1}{4})$.
2. [43] What is the least common multiple of 4004 and 26026?
3. [48] You row upstream at 3 miles per hour and return on the same route at 6 miles per hour. What was your average speed in miles per hour for the whole trip?
4. [48] For what value of n is it true that $3^1 \cdot 3^2 \cdot 3^3 \cdots 3^n = 3^{253}$?
5. [49] A group of students takes a test and the average score is 74. If one more student had taken the test and scored 100, the average would have been 74.5. How many students took the test?
6. [49] Triangle ABC has $AC = 15$, $BC = 13$, and $AB = 4$. What is the length of the altitude from C to the extension of AB?
7. [49] Each of Alice, Bob, Carol, and Don took a test. Each of them answered at least one question correctly, and altogether they answered 67 questions correctly. Alice had more correct answers than anyone else. Bob and Carol together answered 43 questions correctly. How many correct answers did Don have?
8. [48] During the lunch break from 12:00 to 1:00, Bill eats, then checks his messages, then goes to the restroom, then talks to a friend. Each activity after the first takes half as much time as the preceding activity. There are no intervening time intervals. At what time did Bill finish checking his messages?
9. [49] Suppose S is a set of positive integers, each of which is less than 25, such that no two elements of S have a common divisor greater than 1. What is the largest possible number of elements in S?
10. [47] What is the minimal number of positive divisors of x if $x > 1$, and x, $x^{5/6}$, and $x^{7/8}$ are all integers?

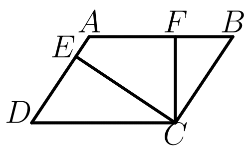
11. [33] What is the number of integers n for which $(7n+15)/(n-3)$ is an integer?

12. [40] A quadratic polynomial f satisfies $f(x) \geq 1$ for all x, $f(2) = 1$, and $f(3) = 3$. What is $f(5)$?

13. [43] Suppose 28 consecutive positive integers have the property that the first one and the last one are both perfect squares. List all possible values for the first of the 28 numbers.

14. [47] Tom, Dick, and Harry were playing tennis. After each game, the loser was replaced by the third person. At the end of the day, Tom had played 27 games, and Dick played 13 games. How many games did Harry play?

15. [33] In parallelogram $ABCD$, $CF \perp AB$ and $CE \perp AD$. If $CF = 2$, $CE = 4$, and FB is one fifth of AB, what is the ratio of the area of quadrilateral $AFCE$ to that of parallelogram $ABCD$?



16. [39] In octagon $ABCDEFGH$, sides AB, BC, CD, DE, and EF are part of a regular decagon, while sides FG, GH, and HA are part of a regular hexagon. What is the degree measure of angle CAH?

17. [39] What is the smallest possible number of students in a class if the percent of girls is less than 50 but at least 47?

18. [30] List all solutions x in $[0, 2]$ to $\sin^2(x^2) = \frac{1}{4}$, where x is in radians.

19. [38] List all real solutions x of the equation

$$(x+2)^{35} + (x+2)^{34}(x-1) + (x+2)^{33}(x-1)^2 + \cdots + (x-1)^{35} = 0.$$

20. [43] Points A, C, and D lie on a circle. Point B lies outside the circle such that B, D, and C are collinear with D between B and C, and BA is tangent to the circle. If $AB = 2$, $AC = 3$, and $BD = 1$, what is the area of triangle ABC?

21. [18] What is the solution set for $x > 0$ of $\dfrac{1}{x+\sqrt{x}} + \dfrac{1}{x-\sqrt{x}} \leq 1$?

22. [15] A right circular cone has height equal to radius. What is the ratio of its volume to that of a cube inscribed inside it, with the base of the cube lying on the base of the cone? Be sure to simplify your answer, using the binomial theorem if relevant.

23. [40] Rectangle $ABCD$ has side AB of length 8, and side AD of length 3. A point particle is ejected into the rectangle from A at a 45 degree angle, and reflects off sides with angle of reflection equal to angle of incidence. Write the ordered pair (V, n), where V is the letter of the vertex that it next hits, and n is the number of sides that it hits before hitting that vertex. (The vertex from which it was ejected does not count in the total.)

24. [34] $ABCD$ is a rectangle in which the length AB minus the length AD equals 10. Inside $ABCD$ is a square $WXYZ$ with sides parallel to those of the rectangle, and W closest to A, and X closest to B. The total of the areas of the trapezoids $XBCY$ and $AWZD$ is 1000, while the total area of the trapezoids $ABXW$ and $ZYCD$ is 400. What is the area of the square $WXYZ$?

25. [35] Let S denote the locus of all points P in the plane such that as a point Q moves along the line segment from $(0, 0)$ to P, the distance from Q to $(2, 0)$ decreases throughout the segment. What is the area of S?

26. [35] How many 10-digit strings of 0's and 1's are there that do not contain any consecutive 0's?

27. [38] Let BE be a median of triangle ABC, and let D be a point on AB such that $BD/DA = 3/7$. What is the ratio of the area of triangle BED to that of triangle ABC?

28. [28] You write five letters to different people, and address the corresponding envelopes. In how many ways can the letters be placed in the envelopes, with one letter in each envelope, so that none of them is in the correct envelope?

29. [35] Determine all ordered pairs (a, b) of positive integers with $a \leq b$ such that $(a + \frac{6}{b})(b + \frac{6}{a}) = 25$.

30. [14] An equilateral triangle in the first quadrant has vertices at the points $(0, 0)$, $(x_1, 4)$, and $(x_2, 11)$. What is the ordered pair (x_1, x_2)?

31. [26] The diagonals of a parallelogram partition it into four triangles. Let G be the centroid of one of the triangles, and let T be a triangle formed by G and two vertices of the parallelogram. What is the largest possible ratio of the area of T to that of the parallelogram?

32. [20] How many ordered 4-tuples of nonnegative integers (a, b, c, d) satisfy $a + b + c + d \leq 15$?

33. [18] List all 4-digit numbers $n = \underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$ such that n has distinct digits and $a \cdot n$, $b \cdot n$, $c \cdot n$, and $d \cdot n$ are 5-digit numbers whose last (rightmost) digit is d, c, b, and a, respectively.

34. [10] Out of all relatively prime integers a and b, what is the largest possible value of the greatest common divisor of $a + 201b$ and $201a + b$?

35. [24] Each of two urns contains N balls. All balls are either red or black, and each urn contains at least one red ball and one black ball. You randomly select an urn and a ball from it, and then put the ball back. Then you do this again. What is the smallest value of N for which it is possible that the probability that you chose two red balls from the first urn equals the probability that you chose two red balls or two black balls from the second urn?

36. [15] Let $f(x) = x^2 + 10x + 20$. List all real values of x for which $f(f(f(f(x)))) = 0$.

37. [0] In the standard version of tic-tac-toe, two players X and O alternately fill in a 3×3 board with their symbols, with X moving first. The game stops when one of the two players has three of his/her symbols in the same row, column, or diagonal, or all squares are filled. How many ways of filling in the entire board with X's and O's can be the end result of a valid game of tic-tac-toe?

38. [5] Find the sum of all positive integers n with no more than 3 digits for which the number obtained as the last three digits of n^2 equals n.

39. [8] In triangle ABC, P lies on AB with $\frac{AP}{PB} = \frac{1}{2}$, and N lies on AC with $\frac{AN}{NC} = 3$. Let M be the midpoint of BC, and G the intersection of lines AM and PN. What is the ratio $\frac{AG}{GM}$?

40. [8] What is the value of $xy + yz + zx$ if $x^2 + xy + y^2 = 2$, $y^2 + yz + z^2 = 1$, and $z^2 + zx + x^2 = 3$, with x, y, and z all positive?

SOLUTIONS TO 2014 CONTEST

1. 6. It is $\frac{1}{2} \cdot 12$.
2. 52052. One is $2^2 1001$ and the other $2 \cdot 13 \cdot 1001$, so the lcm is $2^2 \cdot 13 \cdot 1001$.
3. 4. If the one-way distance is D miles, the time taken is $\frac{D}{3} + \frac{D}{6}$. Set this equal to $\frac{2D}{s}$ to find the speed s. Get $\frac{1}{2} = \frac{2}{s}$.
4. 22. We must have $n(n+1) = 506$, which is most easily solved by inspection.
5. 51. If P is the total number of points scored and n the number of students, then $P = 74n$ and $P + 100 = (74.5)(n+1)$. Subtracting yields $100 = 74.5 + \frac{1}{2}n$, and so $n = 2 \cdot 25.5$.
6. 12. If h is the desired altitude, and x the length of the segment from B to where the altitude hits, then $x^2 + h^2 = 13^2$ and $(4+x)^2 + h^2 = 15^2$. Subtracting yields $8x + 16 = 56$, so $x = 5$ and then $h = 12$.
7. 1. Since one of Bob and Carol had at least 22, Alice must have had at least 23, so the three of them together had at least 66. Thus Don answered at most 1, and since everyone had at least one, he had exactly one.
8. 12:48. The first and second activities took 4 times as long as the third and fourth activities, respectively. Thus 1/5 of the hour was spent on the last two.
9. 10. The largest possible S cannot contain any numbers divisible by distinct primes p and q, because if it did, a larger set could be obtained by replacing this number by p and q. Thus all numbers greater than 1 in the largest possible S must be prime powers, and we might as well choose them to be primes. Thus $S = \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23\}$ is as large as possible.

10. 25. x must be a 6th power and an 8th power. Hence x is a 24th power. The smallest number of factors would be if $x = p^{24}$ for some prime p, and this has 25 divisors.

11. 18. Write it as $7 + \frac{36}{n-3}$. Then $n - 3$ can be any divisor of 36. Since 36 has 9 positive divisors, it has 18 divisors among all integers.

12. 19. We must have $f(x) = a(x-2)^2 + 1$ with $3 = a + 1$. Thus $f(5) = 2(5-2)^2 + 1$.

13. 9 and 169. (Must have both.) $a = n^2$ and $a + 27 = m^2$. Thus $27 = (m-n)(m+n)$. Since 27 can be factored as $3 \cdot 9$ or $1 \cdot 27$, we can have $2n = 9 - 3$ or $27 - 1$.

14. 14. Since every Tom-Harry game is followed by a game involving Dick, they played each other at most 14 times, and so, considering also the times they played against Dick, each of them could have played at most 27 games. Since Tom played 27 games, he did play Harry 14 times (and Dick 13 times), and Harry never played against Dick. In fact, Tom won every game, except possibly the last. (He might have been tired.)

15. 1/2. Let $BF = s$, so $AF = 4s$. Triangle DEC has sides twice those of triangle BFC. Our desired ratio equals
$$\frac{|ABCD| - |FBC| - |CED|}{|ABCD|} = 1 - \frac{s + 4s}{10s} = \frac{1}{2}.$$

16. 114. The desired angle equals $\angle CAF + \angle FAH$. FAH is half of an interior angle of a regular hexagon, hence equal 60 degrees. Angle BAF is half of an interior angle of a regular decagon, hence equals 72 degrees. Triangle ABC is isosceles with vertex angle 144 degrees. Thus angle BAC is 18 degrees, and $CAF = BAF - BAC = 54$.

17. 17. To get the fraction of girls closer to 1/2, there should be an odd number of students in the class. If there are $2n+1$ students,

then we want $\frac{1}{2} - \frac{n}{2n+1} \le \frac{3}{100}$. This reduces to $6(2n+1) \ge 100$ or $2n+1 \ge 17$. Indeed $8/17 \approx .4706$.

18. $\frac{1}{6}\sqrt{6\pi}$, $\frac{1}{6}\sqrt{30\pi}$, $\frac{1}{6}\sqrt{42\pi}$. (Must have all.) Must have $x^2 = \pi/6$, $5\pi/6,\ 7\pi/6,\ 11\pi/6, \ldots$ with $x^2 \le 4$ and $x \ge 0$. Note that $7\pi/6 < 4 < 11\pi/6$.

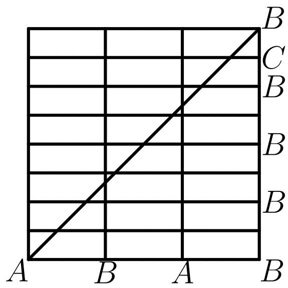
19. $-1/2$. The equation is not satisfied by $x = 1$, so dividing by $(x-1)^{35}$ and letting $q = \frac{x+2}{x-1}$, the equation becomes $q^{35} + q^{34} + \cdots + 1 = 0$. Multiplying by $(q-1)$ implies $q^{36} = 1$ and so $q = -1$. Thus $x + 2 = -x + 1$.

20. $\frac{3}{4}\sqrt{15}$. Since $BC \cdot BD = AB^2$, we obtain $BC = 4$. By Heron's Theorem, the desired area is $\sqrt{\dfrac{9}{2} \cdot \dfrac{1}{2} \cdot \dfrac{3}{2} \cdot \dfrac{5}{2}}$.

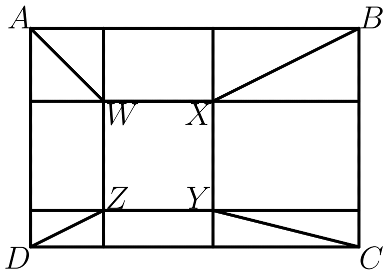
21. $(0,1) \cup [3,\infty)$ or $x < 1$ or $x \ge 3$. It simplifies to $2/(x-1) \le 1$. If $x > 1$, then $x \ge 3$. If $x < 1$, then $x \le 3$.

22. $\frac{\pi}{12}(10 + 7\sqrt{2})$. Let h denote the height of the cone, and s the side of the square. The top of the cube will meet the cone at a level where the radius of the cone is $s/\sqrt{2}$. Thus by similar triangles, $\frac{h-s}{s/\sqrt{2}} = 1$. Thus $h = s(2+\sqrt{2})/2$, and the ratio of the volumes is
$$\frac{\pi h^3/3}{s^3} = \frac{\pi}{24}(2+\sqrt{2})^3 = \frac{\pi}{24}(20 + 14\sqrt{2}) = \frac{\pi}{12}(10 + 7\sqrt{2}).$$

23. $(B, 9)$. In the diagram below, each rectangle is 8×3. The horizontal lines are alternately (Bot,Top,...), while the vertical lines are alternately Left,Right,Left,Right. It first hits a vertex on the Bottom Right corner, and its reflections off sides correspond to crossing either a horizontal or vertical line in the diagram.



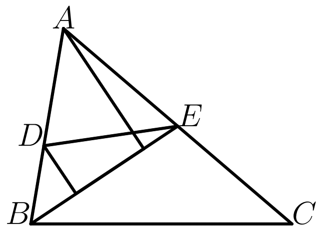
24. 3600. The trapezoids contain equal triangles from the two corners, as indicated below. Thus $1000-400$ equals the sum of the areas of the rectangles to the left and right of $WXYZ$ minus the sum of the areas of the rectangles above and below $WXYZ$, and this is $s(AB-AD)$, where s denotes the side of the square. Thus $s=600/10$.



25. π. If Q moves along a line from $(0,0)$, the distance from $(2,0)$ to Q decreases until Q gets to the point where a perpendicular from $(2,0)$ to the line intersects the line. This extreme point lies on the circle whose diameter is the segment from $(0,0)$ to $(2,0)$. Thus S consists of all points in a circle of radius 1, and hence has area π.

26. 144. Let s_n denote the number of such n-bit sequences. Then $s_1=2$ and $s_2=3$. We have $s_n=s_{n-1}+s_{n-2}$ since a sequence ends either with 1 preceded by any allowable sequence or with 10 preceded by anything. Thus the values of s_n are 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, which are Fibonacci numbers.

27. 3/20. We have $BD/AB = 3/10$, which equals the ratio of the indicated altitude of triangle BDE to that of triangle BAE. Since their bases are the same, that is also the ratio of their areas. Since BE is a median, the areas of triangles BAE and BCE are equal. Thus the area of BDE is 3/20 times that of ABC.

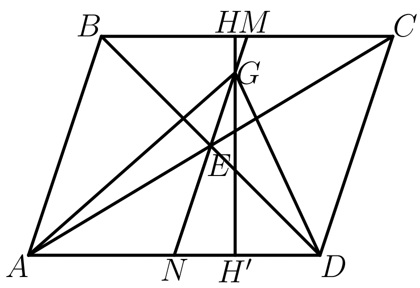


28. 44. Let N be the number of such ways having letter 1 in envelope 5, and assume this has been done. Then the answer will be $4N$. If also letter 5 goes into envelope 1, then the others can be arranged either as 342 or 423. Alternatively, the number of ways in which letter 5 is in envelope 2, 3, and 4 are equal. Say it goes into envelope 2. Then envelopes 1, 3, and 4 can have either letters $(2,4,3)$ or $(3,4,2)$ or $(4,2,3)$. Thus $N = 2 + 3 \cdot 3$.

29. $(1,4)$, $(2,2)$, $(1,9)$, $(3,3)$. (Must have all.) The given equation reduces to $0 = (ab)^2 - 13ab + 36 = (ab-4)(ab-9)$. Hence (a,b) can only be as claimed, and these all work.

30. $(6\sqrt{3}, \sqrt{3})$. The complex number $x_2 + 11i$ equals $(\cos(\pi/3) + i\sin(\pi/3)) \cdot (x_1 + 4i)$. This leads to the equations $\frac{1}{2}x_1 - 2\sqrt{3} = x_2$ and $2 + \frac{\sqrt{3}}{2}x_1 = 11$, which has the claimed solution.

31. 5/12. In the diagram below, M and N are midpoints, and HH' is an altitude. We have
$$\frac{GAD}{ABCD} = \frac{GH'}{2HH'} = \frac{GE + EN}{2MN} = \frac{\frac{2}{3}EM + EM}{4EM} = \frac{5}{12}.$$
It is easily checked that this is the largest triangle.



32. 3876. Let $A = a+1$, etc., so we want positive integers A, B, C, D such that $A+B+C+D \leq 19$. This can be thought of as placing four dividers in the 19 spaces after the numbers $1, \ldots, 19$, with the first divider coming after A, the next one after $A + B$, etc. Thus the answer is $\binom{19}{4}$.

33. 6284 and 6824. (Must have both.) The last digit of d^3 is d, since $d \cdot d$ ends with a, and $a \cdot d$ ends with d. Therefore, $d = 4$ and $a = 6$. $[\![(d = 9, a = 1)$ would not make $a \cdot n$ a 5-digit number, and $d = 0$, 1, 5, or 6 would not make d and a distinct.$]\!]$ Since the last digit of n is even, so are b and c, and they can't be 0. Therefore $n = 6284$ and 6824 are the only two possibilities, and both of them work.

34. 40400. (Half credit for $201^2 - 1$.) Let $m = 201a + b$ and $n = a + 201b$, and let d denote the gcd of m and n. Then d also divides $201m - n = (201^2 - 1)a$, and similarly d divides $(201^2 - 1)b$. Therefore, since a and b are relatively prime, d divides $201^2 - 1$. On the other hand, if $a = 201^2 - 201 - 1$ and $b = 1$, then $m = (201^2 - 1)(201 - 1)$ and $n = 201^2 - 1$, which achieves $201^2 - 1$ as the gcd.

35. 7. Let r_1 and r_2 denote the number of red balls in the two urns, and b_1 and b_2 the number of black balls. Then $r_1 + b_1 = N = r_2 + b_2$. The probability that you chose two red balls from the first urn is $(\frac{r_1}{2N})^2$, while the probability that you chose two red balls or two black balls from the second urn is $(\frac{r_2}{2N})^2 + (\frac{b_2}{2N})^2$.

Thus we must have $r_1^2 = r_2^2 + b_2^2$, for which the smallest solution is $r_1 = 5$, $r_2 = 3$, and $b_2 = 4$.

36. $\pm 5^{1/16} - 5$. (Half-credit if missing the $\pm$.) Note that $f(x) = (x+5)^2 - 5$ or equivalently $f(x) + 5 = (x+5)^2$. Thus

$$f(f(x)) = (f(x)+5)^2 - 5 = (x+5)^4 - 5,$$

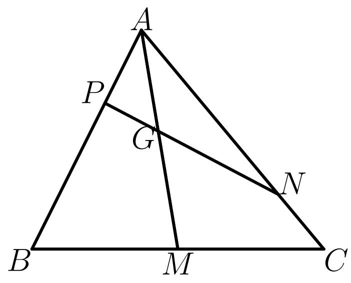
and similarly $f(f(f(f(x)))) = (x+5)^{16} - 5$. To have $f(f(f(f(x)))) = 0$, we must have $x + 5 = \pm 5^{1/16}$.

37. 78. The total number of ways of filling in the board with five X's and 4 O's is $\binom{9}{4} = 126$. Of these, the ones that cannot be the end of a real game are those in which O has a winning position, since X has the ninth move. Note that a configuration in which O does not have a winning position could perhaps have been a configuration in which X could have won prior to the final move, but in such cases there is a sequence of moves in which X did not win until the last move. There are 8 rows/columns/diagonals which could contain three O's, and for each of these, 6 positions where the other O could be placed. None of these 48 positions could be the position at the end of the game. Thus there are $126 - 48$ possible full boards at the end of the game.

38. 1002. We must have $n(n-1) \equiv 0 \mod 2^3 5^3$. Note that $n = 1$ works, and if n works with $1 < n < 1000$, then so does $1001 - n$. Also, n works if and only if $n(n-1)$ is divisible by 2^3 and by 5^3. Since n and $n-1$ are relatively prime and less than 1000, this can happen if and only if one of n and $n-1$ is divisible by 2^3 and the other by 5^3. If n is divisible by 5^3, then $n \in \{5^3, 2 \cdot 5^3, \ldots, 7 \cdot 5^3\}$. This is a complete set of nonzero residues mod 8, and so exactly one of them satisfies $i \cdot 5^3 \equiv 1 \mod 8$. Then $n - 1$ is divisible by 8 if and only if n equals this $i \cdot 5^3$. Thus there is exactly one n in the desired range with n divisible by 5^3 and $n-1$ divisible by 2^3. Similarly there is exactly one n with n divisible by 2^3 and $n-1$ divisible by 5^3. By the earlier observation, these two values of n must satisfy

that their sum is 1001. Including also $n = 1$ yields the claim. (In fact, the numbers are 1, 625, and 376.)

39. 6/7. Using $|-|$ for area, we have
$$\frac{|APG|}{|ABM|} = \frac{AP}{AB} \cdot \frac{AG}{AM}.$$
Since $|ABM| = \frac{1}{2}|ABC|$ and $\frac{AP}{AB} = \frac{1}{3}$, we obtain $\frac{|APG|}{|ABC|} = \frac{1}{6} \cdot \frac{AG}{AM}$. Similarly $\frac{|AGN|}{|ABC|} = \frac{3}{8} \cdot \frac{AG}{AM}$. Also, $\frac{|APN|}{|ABC|} = \frac{AP}{AB} \cdot \frac{AN}{AC} = \frac{1}{4}$. Thus $(\frac{1}{6} + \frac{3}{8})\frac{AG}{AM} = \frac{1}{4}$, so $\frac{AG}{AM} = \frac{6}{13}$, and $\frac{AG}{GM} = \frac{6}{7}$.



40. $\frac{2}{3}\sqrt{6}$. Note that, by the Law of Cosines, if triangle AOB has angle AOB equal to 120 degrees, and $AO = x$ and $OB = y$, then $x^2 + xy + y^2 = AB^2$. Suppose that x, y, and z satisfy the three equations. From a point O draw segments OA, OB, and OC making 120 degree angles with one another of lengths x, y, and z, where x, y, and z satisfy the given equations. Then triangle ABC will have sides $\sqrt{2}$, 1, and $\sqrt{3}$. This is a right triangle with area $\frac{1}{2}\sqrt{2}$. The areas of the three subtriangles are $xy\sqrt{3}/4$, $yz\sqrt{3}/4$, and $zx\sqrt{3}/4$, since $\sin(120) = \sqrt{3}/2$. Thus $(xy + yz + zx)\sqrt{3}/4 = \frac{1}{2}\sqrt{2}$.