GEODESICS IN THE CONFIGURATION SPACES OF TWO POINTS IN $\mathbb{R}^n$

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Abstract. We determine explicit formulas for geodesics (in the Euclidean metric) in the configuration space of ordered pairs (x, x') of points in $\mathbb{R}^n$ which satisfy $d(x, x') \geq \varepsilon$. We interpret this as two or three (depending on the parity of n) geodesic motion-planning rules for this configuration space. In the associated unordered configuration space, we need not prescribe that the points stay apart by ε. For this space, with a Euclidean-related metric, we show that geodesic motion-planning rules correspond to ordinary motion-planning rules on RP^{n-1}.

1. Results

Recently David Recio-Mitter ([5]) introduced the notion of geodesic complexity, which is an analogue of Farber's topological complexity ([3]), but requires that paths be minimal geodesics. This is a useful requirement for efficient motion-planning algorithms. In [5] and [2], the geodesic complexity of several spaces was determined.

Configuration spaces are of central importance in topological robotics, since they model the situation of several robots moving throughout a region. In this paper, we first consider the case of two distinguished points (or balls) moving in $\mathbb{R}^n$. We obtain explicit formulas for the geodesics and optimal geodesic motion-planning rules. We also consider two indistinguishable points moving in $\mathbb{R}^n$, and show that geodesic motion-planning rules for these correspond to ordinary motion-planning rules in real projective space RP^{n-1}.

Let $F(\mathbb{R}^n, 2)$ denote the ordered configuration space of two distinct points in $\mathbb{R}^n$. It is a subspace of $\mathbb{R}^{2n}$ and is given the Euclidean metric. This space is not geodesically complete. For example, there is no geodesic from $((1, \overline{0}), (\overline{0}, 1))$ to $((-1, \overline{0}), (\overline{0}, -1))$

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since the linear path $\sigma(t) = ((1-2t,\bar{0}),(\bar{0},1-2t))$ has $\sigma(\frac{1}{2}) \notin F(\mathbb{R}^n,2)$, but there are paths in $F(\mathbb{R}^n,2)$ between these points arbitrarily close to σ. By "geodesic," we will always mean "minimal geodesic."

For a positive number ε, we consider the subspace of $F(\mathbb{R}^n,2)$ consisting of points (x,x') for which $d(x,x') \ge \varepsilon$. By scaling, we may assume $\varepsilon = 2$, and define

$$F_0(\mathbb{R}^n,2) = \{(x,x') \in F(\mathbb{R}^n,2) :\ d(x,x') \ge 2\}.$$

This can be viewed as the space of ordered pairs of disjoint open unit balls in $\mathbb{R}^n$. Note that $F_0(\mathbb{R}^n,2)$ is a manifold with boundary ∂F_0 consisting of points of the form $(x-u,x+u)$ with $\|u\|=1$. In the following theorem, we give explicit formulas for geodesics in $F_0(\mathbb{R}^n,2)$ between any two points.

Theorem 1.1. *Let $P=(a,a')$ and $Q=(b,b')$ be points of $F_0(\mathbb{R}^n,2)$. Let*

$$h=(a'-a)/2, \quad k=(b'-b)/2, \quad A=(a'+a)/2, \quad B=(b'+b)/2.$$

Let $\delta = \min\{d(tb+(1-t)a, tb'+(1-t)a') :\ 0 \le t \le 1\}$, the minimal distance between the two components of the linear path between P and Q.

a. *If $\delta \ge 2$, the linear path from P to Q is the unique geodesic between P and Q in $F_0(\mathbb{R}^n,2)$.*

b. *If $0<\delta\le 2$, there is a unique geodesic in $F_0(\mathbb{R}^n,2)$ from P to Q. It is the path composition $\ell_1\sigma\ell_2$, where ℓ_1 is the linear path from P to $C_0=(x-u,x+u)$, σ the geodesic in ∂F_0 from C_0 to $C_1=(y-v,y+v)$, described in Proposition 1.4, and ℓ_2 the linear path from C_1 to Q. Here u and v are unique unit vectors in $\mathbb{R}^n$ satisfying*

$$h\cdot u = 1 \text{ and } k\cdot v = 1 \tag{1.2}$$

with minimal $\|u-v\|$. Let β be the angle between this u and v with $0\le\beta<\pi$. Then

$$x = \frac{\beta A + S_0 B + S_1 A}{\beta + S_0 + S_1}, \quad y = \frac{\beta B + S_0 B + S_1 A}{\beta + S_0 + S_1}, \tag{1.3}$$

where $S_0 = \sqrt{\|h\|^2-1}$ and $S_1 = \sqrt{\|k\|^2-1}$. If $\beta=0$, then $C_0=C_1$, $\delta=2$, and $\ell_1\ell_2$ is the linear path from P to Q. When

$\delta = 2$, the linear path in (a) can also be obtained by the method of (b).

c. *If $\delta = 0$, then h and k are parallel in opposite directions, and conversely. In this case, the unit vector solutions u, v of (1.2) with minimal $\|u - v\|$ are*

$$u = \frac{h}{\|h\|^2} + \frac{S_0}{\|h\|}w, \quad v = \frac{k}{\|k\|^2} + \frac{S_1}{\|k\|}w,$$

where w ranges over the set of all points satisfying $h \cdot w = 0$ and $w \cdot w = 1$. The geodesics from P to Q are paths as described in (b) for each of these pairs u, v, using (1.3) and Corollary 2.5.

We will refer to these as type (a), (b), or (c) paths or Situations. In Proposition 2.4 and Corollary 2.5 we give explicit formulas for u, v, and β in terms of h and k.

The geodesics in ∂F_0 to which we just referred are described in the following result.

Proposition 1.4. *Let u and v be unit vectors in $\mathbb{R}^n$, and α be the angle from u to v with $0 \le \alpha < \pi$. For $0 \le t \le 1$, let*

$$u(t) = \frac{\sin((1-t)\alpha)u + \sin(t\alpha)v}{\sin\alpha}$$

if $\alpha > 0$. If $\alpha = 0$, then $u(t) = u = v$ for all t. For $x, y \in \mathbb{R}^n$, the unique geodesic in ∂F_0 from $(x - u, x + u)$ to $(y - v, y + v)$ is the curve

$$\sigma(t) = ((1-t)x + ty - u(t), (1-t)x + ty + u(t)).$$

Its length is

$$\sqrt{2(\|x - y\|^2 + \alpha^2)}. \tag{1.5}$$

Recall that the geodesic complexity $\mathrm{GC}(X)$ is the smallest k such that $X \times X$ can be partitioned into ENRs $E_0, \ldots, E_k$ such that on each E_i there is a continuous map s_i from E_i to the free path space PX such that $s_i(x_0, x_1)$ is a geodesic from x_0 to x_1. ([5]) The topological complexity $\mathrm{TC}(X)$ is defined similarly without requiring that the paths be geodesics.([3]) Our second result is the determination of $\mathrm{GC}(F_0(\mathbb{R}^n, 2))$.

Theorem 1.6. *For $n \ge 2$,*

$$\mathrm{GC}(F_0(\mathbb{R}^n, 2)) = \mathrm{TC}(F_0(\mathbb{R}^n, 2)) = \mathrm{TC}(S^{n-1}) = \begin{cases} 1 & n \text{ even} \\ 2 & n \text{ odd.} \end{cases}$$

The unordered configuration space $C(\mathbb{R}^2,2)$ is the quotient of $F(\mathbb{R}^n,2)$ by the involution which reverses the order of the two points. Points of $C(\mathbb{R}^n,2)$ are sets $\{a,a'\}$ with $a,a'\in\mathbb{R}^n$. Surprisingly, $C(\mathbb{R}^n,2)$ is, in some sense, easier for these considerations than $F(\mathbb{R}^n,2)$.

Theorem 1.7. *With d denoting the Euclidean metric in $\mathbb{R}^n\times\mathbb{R}^n$, defining*

$$d_U(\{a,a'\},\{b,b'\})=\min(d((a,a'),(b,b')),\ d((a,a'),(b',b)))$$

gives a metric on $C(\mathbb{R}^n,2)$ which has linear geodesics between any two points.

So, we need not bother with the intricacies for geodesics in $F(\mathbb{R}^n,2)$ caused by the need to keep points at least a certain distance apart. The space $C(\mathbb{R}^n,2)$ has the homotopy type of RP^{n-1}, and so the following result, which we prove in Section 4, may not be surprising. The proof will show that the geodesics in $C(\mathbb{R}^n,2)$ are obtained from not-necessarily-geodesic paths in RP^{n-1}.

Theorem 1.8. *For $n\ge 2$, $\mathrm{GC}(C(\mathbb{R}^n,2))=\mathrm{TC}(RP^{n-1})$.*

By [4], $\mathrm{TC}(RP^n)$ equals the immersion dimension of RP^n unless $n=1$, 3, or 7, but this does not enter into our proof.

In Section 5, we consider a different metric on $F(\mathbb{R}^n,2)$ in which it is geodesically complete, and discuss geodesics in that metric.

2. Proof of Theorem 1.1

The following proof of Proposition 1.4 benefited from ideas of David L. Johnson.

Proof of Proposition 1.4. With $u\cdot v=\cos\alpha$ and $u\cdot u=1=v\cdot v$, $u(t)=c_0u+c_1v$ is obtained by solving $u\cdot(c_0u+c_1v)=\cos(t\alpha)$ and $(c_0u+c_1v)\cdot(c_0u+c_1v)=1$. We have

$$\begin{aligned}\sigma'(t)\ &=\ (-x+y-\tfrac{\alpha}{\sin\alpha}(-\cos((1-t)\alpha)u+\cos(t\alpha)v),\\ &\qquad -x+y+\tfrac{\alpha}{\sin\alpha}(-\cos((1-t)\alpha)u+\cos(t\alpha)v)).\end{aligned}$$

Expanding $\cos(\alpha-t\alpha)$ yields $\|-\cos((1-t)\alpha)u+\cos(t\alpha)v\|^2=\sin^2\alpha$, and hence $\|\sigma'(t)\|^2=2(\|x-y\|^2+\alpha^2)$, which implies the claim about the length of the curve.

A constant-speed curve σ with σ'' orthogonal to the surface is a geodesic. We have $\sigma''(t)=\frac{\alpha^2}{\sin\alpha}(\sin((1-t)\alpha)u+\sin(t\alpha)v,-(\sin((1-t)\alpha)u+\sin(t\alpha)v))$.

The surface ∂F_0 is parametrized by $X(x,u) = (x-u, x+u)$ with $x \in \mathbb{R}^n$ and $u \in S^{n-1}$. Then $\sigma''(t)$ is orthogonal to the x-directions and is orthogonal to the spherical parameter u since it is a multiple of the radius vector at each point.

Since $\sigma'(0) = (-x+y-\frac{\alpha}{\sin\alpha}(-\cos(\alpha)u+v), -x+y+\frac{\alpha}{\sin\alpha}(-\cos(\alpha)u+v))$, every tangent direction from the initial point $(x-u, x+u)$ is obtained for one of our geodesics, showing that they are unique. ■

The following lemma will be very important to our analysis.

Lemma 2.1. *Let $(a,a') \in F_0(\mathbb{R}^n, 2)$, and let $h = (a'-a)/2 \in \mathbb{R}^n$. If u is a unit vector in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, the segment between (a,a') and $(x-u, x+u)$ lies in $F_0(\mathbb{R}^n, 2)$ iff $h \cdot u \geq 1$.*

Proof. We require that for $t \in [0,1]$

$$d(ta + (1-t)(x-u), ta' + (1-t)(x+u)) \geq 2.$$

Halving and squaring, this becomes

$$\begin{aligned} 1 &\leq \|th + (1-t)u\|^2 \\ &= t^2\|h\|^2 + 2t(1-t)h \cdot u + (1-t)^2. \end{aligned}$$

By assumption, $\|h\|^2 \geq 1$, so this quadratic function $f(t)$ satisfies $f(0) = 1$ and $f(1) \geq 1$. It is ≥ 1 for all $t \geq 0$ iff $f'(0) \geq 0$. Since $f'(0) = -2 + 2h \cdot u$, the result follows. ■

Because of Lemma 2.1, paths of the form $\ell_1\sigma\ell_2$ in Theorem 1.1(b) exist as long as $h \cdot u \geq 1$ and $k \cdot v \geq 1$ for unit vectors u and v, for any x and y. The proof of Theorem 1.1 will show that minimal length of such paths is achieved when $h \cdot u = 1 = k \cdot v$ and $\|u - v\|$ is minimized. The following result relates intersections of the hyperplanes $h \cdot u = 1$ and $k \cdot u = 1$ inside and on the unit sphere to the value of δ in Theorem 1.1, which in turn determines the types of geodesics.

Proposition 2.2. *Let h, k, and δ be as in Theorem 1.1. Let $H = \|h\|^2 \geq 1$, $K = \|k\|^2 \geq 1$, and $D = h \cdot k$.*

- *If $\min(H,K) \leq D$, then $\delta^2 = 4\min(H,K) \geq 4$, so $\delta \geq 2$.*

- *If* $\min(H,K) \ge D$ *and* $h \ne k$*, then*

$$\delta^2 = 4(HK - D^2)/(H + K - 2D).$$

In this case, regarding solutions of $h \cdot u = 1$ *and* $k \cdot u = 1$*, we have*

i. *There exist solutions with* $\|u\| < 1$ *(and more than one solution with* $\|u\| = 1$*) iff* $\delta > 2$*.*

ii. *There exists a unique solution with* $\|u\| = 1$ *iff* $\delta = 2$*.*

iii. *There exist no solutions with* $\|u\| \le 1$ *iff* $\delta < 2$*.*

Proof. Note that $HK \ge D^2$ by Cauchy-Schwarz, and $H+K > 2D$, since $\|h\|^2+\|k\|^2 > 2\|h\|\,\|k\|\cos\alpha$ when $h \ne k$.

Let $d^2(t) = \|2tk + 2(1-t)h\|^2 = 4(t^2K + (1-t)^2H + 2t(1-t)D)$. Then $\delta^2 = \min(d^2(t) : \ t \in [0,1])$. The minimum of $d^2(t)$ over all $t \in \mathbb{R}$ occurs when $t = t_0 := \frac{H-D}{H+K-2D}$, and has value $4\frac{HK-D^2}{H+K-2D}$. Note that $t_0 \in [0,1]$ iff $\min(H,K) \ge D$. If $\min(H,K) \le D$, then $d^2(t)$ does not have a relative minimum for $0 < t < 1$ so its absolute minimum on $[0,1]$ occurs at an endpoint.

If k is a scalar multiple of h, the result is easily verified, so we assume this is not the case, and have $HK - D^2 > 0$. Now let $u = \alpha h + \beta k + \ell$ with ℓ orthogonal to h and k. If $n = 2$, omit ℓ. The equations $h \cdot u = 1$ and $k \cdot u = 1$ become $\alpha H + \beta D = 1$ and $\alpha D + \beta K = 1$, whose solution is

$$\alpha = \frac{K - D}{HK - D^2}, \qquad \beta = \frac{H - D}{HK - D^2},$$

yielding

$$u \cdot u = \frac{H + K - 2D}{HK - D^2} + \ell \cdot \ell. \tag{2.3}$$

Thus when k is not a scalar multiple of h, we have $\|u\|^2 = \frac{4}{\delta^2} + \|\ell\|^2$. The conclusions follow. By the complementary nature of the cases, it suffices to show implication in one direction. For each hypothesis on δ, the conclusion about $\|u\|$ for solutions is clear. In case (i), the solutions are obtained by varying ℓ. ■

By Proposition 2.2, the next result applies exactly when $\delta \le 2$.

Proposition 2.4. *Let h and k satisfy $\|h\| > 1$ and $\|k\| > 1$. Assume there does not exist u with $h \cdot u = 1 = k \cdot u$ with $\|u\| < 1$. The solutions of $h \cdot u = 1 = k \cdot v$ and $\|u\| = 1 = \|v\|$ with minimal $\|u - v\|$ are*

i. *If k is a scalar multiple of h, then*

$$u = \frac{h}{\|h\|^2} + \frac{S_0}{\|h\|}w, \quad v = \frac{k}{\|k\|^2} + \frac{S_1}{\|k\|}w,$$

where w ranges over the set of vectors satisfying $h \cdot w = 0$ and $w \cdot w = 1$. Here S_0 and S_1 are as in Theorem 1.1.

ii. *If k is not a scalar multiple of h, then there is a unique solution, using notation of Proposition 2.2,*

$$\begin{aligned} u &= \frac{h}{H} + \frac{S_0(k - \frac{D}{H}h)}{\sqrt{HK - D^2}} \\ v &= \frac{k}{K} + \frac{S_1(h - \frac{D}{K}k)}{\sqrt{HK - D^2}}. \end{aligned}$$

Proof. (i.) The vectors u and v lie on the intersections with the unit sphere of parallel hyperplanes. The vector u can be written uniquely as $u = c_0 h + c_1 w$ satisfying $h \cdot u = 1$, $u \cdot u = 1$, $w \cdot w = 1$, and $h \cdot w = 0$. These equations yield the above formula for u, and similarly for v. The vector v closest to u will be the one with the same unit vector w, and for all vectors w, the values of $\|u - v\|$ are the same.

(ii.) Let $\ell = k - \frac{D}{H}h$, which is orthogonal to h. Then v can be written uniquely as $ah + b\ell + m$, with m orthogonal to h and ℓ. The point v closest to the hyperplane $h \cdot u = 1$ will be the one with the largest a. From $v \cdot k = 1$, we deduce $1 = aD + b(HK - D^2)/H$. Use this to eliminate b. Then $v \cdot v = 1$ implies

$$1 = a^2 H + \left(\frac{H(1 - aD)}{HK - D^2}\right)^2 \left(\frac{HK - D^2}{H}\right) + M,$$

where $M = \|m\|^2$. This simplifies to

$$0 = HK\, a^2 - 2Da + 1 - (1 - M)(HK - D^2)/H,$$

which has solution

$$a = \frac{D \pm \sqrt{D^2 + K((1 - M)(HK - D^2) - H)}}{HK}.$$

The maximum of this occurs when $M = 0$ and has $a = (D+\sqrt{(K-1)(HK-D^2)})/HK$. One easily finds b now and obtains the formula for v. The formula for u is obtained similarly. ■

Corollary 2.5. *The angle β in Theorem 1.1(b) satisfies*

$$\cos(\beta) = \frac{(S_0+S_1)\sqrt{HK-D^2}+(1-S_0S_1)D}{HK}.$$

The angle β in Theorem 1.1(c) satisfies

$$\cos(\beta) = \frac{D}{HK} + \frac{S_0S_1}{\sqrt{HK}}.$$

Proof. We compute $\cos(\beta) = u\cdot v$ from Proposition 2.4. ■

Proof of Theorem 1.1. The conclusion of part (a) is immediate. Next we reduce consideration of part (c) to that of part (b).

First note that $\delta = 0$ iff $tb+(1-t)a = tb'+(1-t)a'$ for some $t\in[0,1]$ iff $b'-b$ and $a'-a$ are negative multiples of one another. Then the analysis of type-(b) paths which follows applies, with minor modifications which are discussed below, to all of the pairs u, v obtained in Proposition 2.4(i), listed again in Theorem 1.1(c).

Geodesics in a manifold with boundary are path compositions of geodesics in the manifold and geodesics in the boundary.(e.g., [1].) In our case, this will consist of at most one geodesic in ∂F_0. [[If it were $\ell_1\sigma_1\ell_2\sigma_2\ell_3$, then ℓ_2 would be a line segment connecting two points of ∂F_0. Similarly to the proof of Lemma 2.1, the line segment connecting two points $(x-u, x+u)$ and $(y-v, y+v)$ of ∂F_0 will lie outside $F_0(\mathbb{R}^n, 2)$ unless the two points have $u = v$, in which case it is a line segment lying in ∂F_0. When path-multiplied by an angle-changing geodesic in ∂F_0, the result will not be a geodesic.]] So we need just consider path compositions of the form $\ell_1\sigma\ell_2$.

If $\|h\| = 1$, then $P \in \partial F_0$. By the argument just described, we may then choose ℓ_1 to be the constant path. This is consistent with (1.3) since we would have $S_0 = 0$, $x = A$, and $u = h$. Similarly, if $\|k\| = 1$, the path ℓ_2 may be ignored. Thus we shall assume $\|h\| > 1$ and $\|k\| > 1$, so Proposition 2.4 applies.

With the notation of the theorem, let $\widehat{D}_1$ denote the length of the linear path ℓ_1 in $F_0(\mathbb{R}^n, 2)$ from P to any point $(x-u, x+u)$ with $\|u\| = 1$. This equals

$$\begin{aligned} & \sqrt{\|x-u-a\|^2 + \|x+u-a'\|^2} \\ = & \sqrt{\|a\|^2 + \|a'\|^2 + 2\|x\|^2 + 2 - 4x\cdot A - 4h\cdot u} \\ = & \sqrt{2}\sqrt{\|h\|^2 + 1 + \|x-A\|^2 - 2h\cdot u}. \end{aligned} \tag{2.6}$$

A path $\ell_1\sigma\ell_2$ has length $\widehat{D}_1 + \widehat{D}_3 + \widehat{D}_2$, where $\widehat{D}_3$ is the length of the curved path σ described in Proposition 1.4, and $\widehat{D}_2$ is a formula similar to (2.6) for a linear path ℓ_2 from $(y-v, y+v)$ to Q. Let $D_i = \widehat{D}_i/\sqrt{2}$ and $T = D_1 + D_3 + D_2$. If $h\cdot u = 1 = k\cdot v$ and α is the angle between u and v, the formulas for D_1 and D_2 simplify nicely, and we have

$$T = \sqrt{S_0^2 + \|x-A\|^2} + \sqrt{\|x-y\|^2 + \alpha^2} + \sqrt{S_1^2 + \|y-B\|^2}. \tag{2.7}$$

Setting $\partial T/\partial x_i = 0$ gives

$$\frac{x_i - A_i}{\sqrt{S_0^2 + \|x-A\|^2}} + \frac{x_i - y_i}{\sqrt{\|x-y\|^2 + \alpha^2}} = 0, \tag{2.8}$$

so

$$(x_i - A_i)^2(\|x-y\|^2 + \alpha^2) = (x_i - y_i)^2(S_0^2 + \|x-A\|^2).$$

Summing over i and cancelling yields $\alpha^2\|x-A\|^2 = S_0^2\|x-y\|^2$. Now (2.8) says $\alpha(x-A) = (y-x)S_0$. Similarly $\alpha(y-B) = (x-y)S_1$. Solving these equations yields (1.3), with β replaced by α. This is a consequence of $\partial T/\partial x_i = 0 = \partial T/\partial y_i$ and (1.2).

When x and y are as in (1.3),

$$x - y = \tfrac{(A-B)\alpha}{\alpha+S_0+S_1}, \quad x - A = \tfrac{(B-A)S_0}{\alpha+S_0+S_1}, \quad y - B = \tfrac{(A-B)S_1}{\alpha+S_0+S_1}, \tag{2.9}$$

and we obtain the dramatic simplification

$$T = \sqrt{\|A-B\|^2 + (\alpha + S_0 + S_1)^2},$$

showing clearly that we should choose β to minimize α.

Note that $\beta < \pi$ (so Lemma 1.4 applies), since the only way to have $\beta = \pi$ would be with the hyperplanes $h\cdot u = 1$ and $k\cdot v = 1$ tangent to the unit sphere, and parallel, so $\|h\| = 1 = \|k\|$, which we have removed from our consideration.

We must also consider changes of T caused by changes in u or v. The more general formula for T at any point is

$$\begin{aligned} T &= \sqrt{H+1+\|x-A\|^2-2h\cdot u}+\sqrt{\|x-y\|^2+(\arccos(u\cdot v))^2}\\ &\quad+\sqrt{K+1+\|y-B\|^2-2k\cdot v}. \end{aligned} \tag{2.10}$$

At our claimed critical point, which was derived from $\frac{\partial T}{\partial x_i}=0=\frac{\partial T}{\partial y_i}$, (1.2), and minimal $\|u-v\|$, u and v lie in the h-k plane by Proposition 2.4(ii). Changes in u or v orthogonal to the h-k plane will not affect (2.10) at this point. For Situation (c), each pair u, v is determined by a choice of w. They lie in the h-w plane. The analysis here applies with h-k replaced by h-w.

Letting α denote the angle from v to u, and using $\sqrt{\|x-y\|^2+\alpha^2}$ for the middle term of (2.10), we obtain

$$\begin{aligned} \frac{\partial T}{\partial\alpha} &= \frac{-h\cdot\frac{du}{d\alpha}}{\sqrt{S_0^2+\|x-A\|^2}}+\frac{\alpha}{\sqrt{\|x-y\|^2+\alpha^2}}\\ &= \frac{\alpha+S_0+S_1}{\sqrt{(\alpha+S_0+S_1)^2+\|B-A\|^2}}\left(\frac{-h\cdot\frac{du}{d\alpha}}{S_0}+1\right), \end{aligned} \tag{2.11}$$

incorporating (2.9). We can parametrize $\mathbb{R}^n$ so that $v=(1,\overline{0})$, $u=(\cos\alpha,\sin\alpha,\overline{0})$, and $h=(h_1,h_2,\overline{0})$. We obtain $h\cdot\frac{du}{d\alpha}=-h_1\sin\alpha+h_2\cos\alpha$, so

$$\begin{aligned} S_0^2-(h\cdot\tfrac{du}{d\alpha})^2 &= h_1^2+h_2^2-1-(h_1^2\sin^2\alpha+h_2^2\cos^2\alpha-2h_1h_2\cos\alpha\sin\alpha)\\ &= (h_1\cos\alpha+h_2\sin\alpha)^2-1\\ &= 0. \end{aligned}$$

Noting that $h\cdot\frac{du}{d\alpha}\ge 0$ since $h\cdot u$ had minimal allowable value (1) at our point, we obtain $h\cdot\frac{du}{d\alpha}=S_0$, and so $\frac{\partial T}{\partial\alpha}=0$ in (2.11).

We have shown that the assumption (1.2) leads to the unique critical point of T described in Theorem 1.1(b) when h and k are not parallel, and to any of the claimed points in Situation (c), and Lemma 2.1 says that we must have $h\cdot u\ge 1$ and $k\cdot v\ge 1$. If $h\cdot u=t>1$, corresponding to a larger value of α, then we can find values of x and y that make $\partial T/\partial x_i=0=\partial T/\partial y_i$ with formulas similar to (1.3) except that now $S_0=\sqrt{\|h\|^2+1-2t}$. An analysis similar to the above paragraph will lead to $\partial T/\partial\alpha>0$ for changes in the h-k plane. So points with $h\cdot u>1$ or $k\cdot v>1$ cannot be critical points. We conclude that our critical point is a unique minimum of T in Situation (b), and our claimed points are the only critical points in Situation (c).

Next we justify the next-to-last sentence of part (b) by noting that if $\beta = 0$, so $u = v$ and then clearly $x = y$ in (1.3), and then showing that

$$(x - u, x + u) = \tfrac{S_1}{S_0+S_1} P + \tfrac{S_0}{S_0+S_1} Q,$$

so the unique point where the lines from P and Q meet ∂F_0 is on the line connecting P and Q.

This requires showing that

$$\tfrac{S_0 B + S_1 A}{S_0+S_1} \pm u = \tfrac{S_1}{S_0+S_1}(A \pm h) + \tfrac{S_0}{S_0+S_1}(B \pm k),$$

hence we need to prove

$$u = \frac{S_1 h + S_0 k}{S_0 + S_1}. \tag{2.12}$$

By Proposition 2.4(ii), u is in the h-k plane. We parametrize that plane so that $h = (h_1, h_2)$, $k = (k_1, k_2)$, and $u = (\cos\theta, \sin\theta)$. Since $h \cdot u = 1$, $h_2 = (1 - h_1\cos\theta)/\sin\theta$, so

$$S_0 = \sqrt{h_1^2 + \left(\tfrac{1-h_1\cos\theta}{\sin\theta}\right)^2 - 1} = \frac{\sqrt{\cos^2\theta - 2h_1\cos\theta + h_1^2}}{|\sin\theta|} = \left|\frac{\cos\theta - h_1}{\sin\theta}\right|,$$

and similarly $S_1 = |(\cos\theta - k_1)/\sin\theta|$.

Since θ is the common endpoint of (otherwise disjoint) intervals on which $h_1\cos\theta + h_2\sin\theta \geq 1$ and $k_1\cos\theta + k_2\sin\theta \geq 1$, the derivatives of these expressions must have opposite signs at θ. The derivative of $h_1\cos\theta + h_2\sin\theta$ is

$$-h_1\sin\theta + \tfrac{1-h_1\cos\theta}{\sin\theta}\cos\theta = \tfrac{\cos\theta - h_1}{\sin\theta}.$$

Thus $\cos\theta - h_1$ and $\cos\theta - k_1$ have opposite signs. Thus

$$\frac{S_1 h_1 + S_0 k_1}{S_0 + S_1} = \frac{(\cos\theta - k_1)h_1 - (\cos\theta - h_1)k_1}{-(\cos\theta - h_1) + (\cos\theta - k_1)} = \cos\theta.$$

Similarly, $(S_1 h_2 + S_0 k_2)/(S_0 + S_1) = \sin\theta$, proving (2.12) and hence the next-to-last sentence of part (b) of Theorem 1.1.

Finally, regarding the last sentence of part (b): If $\delta = 2$, there are points $c = (1-t)a + tb$ and $c' = (1-t)a' + tb'$ such that $d(c, c') = 2$. There is a unique unit ball having cc' as a diameter, and the method of (b) will yield ℓ_1 the linear path from (a, a') to (c, c'), σ a constant path, and ℓ_2 the linear path from (c, c') to (b, b'). ■

Proof of Theorem 1.6. Let E_0 denote the set of all $(P,Q) \in F_0(\mathbb{R}^n,2) \times F_0(\mathbb{R}^n,2)$ of type (c) in Theorem 1.1, and E_1 it complement. First we show that the unique geodesics at points (P,Q) of E_1 vary continuously with (P,Q), giving a geodesic motion planning rule on E_1.

For the type-(b) geodesics, the issue is whether u and v, hence β, vary continuously with (P,Q). Small changes in (P,Q) cause small changes in h and k, and hence small changes in u and v of norm 1 satisfying $h \cdot u = 1 = k \cdot v$. If (u,v) has minimal positive $\|u-v\|$ for such vectors, there will be a neighborhood of (u,v) on which this is true. Alternatively, u and v vary continuously with h and k by Corollary 2.5.

The linear paths of type (a) vary continuously with the parameters. By the last two sentences of Theorem 1.1(b), the paths in the intersection of types (a) and (b) agree, and so by the Pasting Lemma, we have a continuous choice of geodesics on E_1.

If n is even, let V be a unit-length vector field on S^{n-1}. In Theorem 1.1(c), let $w = V(\frac{h}{\|h\|})$. This leads to a continuous choice of geodesics on E_0. Hence $\mathrm{GC}(F_0(\mathbb{R}^n,2)) \le 1$ if n is even. Since

$$F_0(\mathbb{R}^n,2) \approx \mathbb{R}^n \times \{x \in \mathbb{R}^n : \|x| \ge 2\} \simeq S^{n-1} \tag{2.13}$$

and TC is a homotopy invariant, we obtain, for n even,

$$\mathrm{GC}(F_0(\mathbb{R}^n,2)) \ge \mathrm{TC}(F_0(\mathbb{R}^n,2)) = \mathrm{TC}(S^{n-1}) = 1 \ge \mathrm{GC}(F_0(\mathbb{R}^n,2)).$$

Hence we have equality.

If n is odd, let V be a unit-length vector field on $S^{n-1} - \{(1,\overline{0})\}$. Then $w = V(\frac{h}{\|h\|})$ in 1.1(c) leads to a continuous choice of geodesics on $E_0 - Z$, where Z is the set of $((a,a'),(b,b')) \in E_0$ such that $a'-a$ and $b'-b$ are scalar multiples of $(1,\overline{0})$. On Z, you could use the geodesics obtained from using $w = (0,1,\overline{0})$ in 1.1(c). Thus $\mathrm{GC}(F_0(\mathbb{R}^n,2)) \le 2$, and since $\mathrm{TC}(S^{n-1}) = 2$, we have equality as in the previous paragraph.

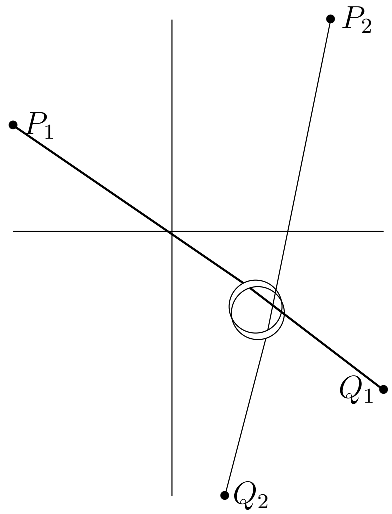
■

3. Examples when $n = 2$

We illustrate two examples of geodesics when $n = 2$.

Let $P = (P_1, P_2) = ((-6, 4), (6, 8))$ and $Q = (Q_1, Q_2) = ((8, -6), (2, -10))$. We have $h = (6, 2)$ and $k = (-3, -2)$. From (1.3), we obtain $x = (3.1596, -2.8468)$ and $y = (3.2474, -3.0927)$. From Proposition 2.4, we obtain $u = (.4622, -.8867)$ and $v = (.3022, -.9533)$, and from Corollary 2.5, $\beta = .1736$. In this example, our path has length 25.2455, whereas the straight line path from P to Q (which is not in $F_0(\mathbb{R}^2, 2)$) has length 25.2190. Although we cannot quite draw the short middle part of the paths, in Figure 3.1 we picture the paths in this example.

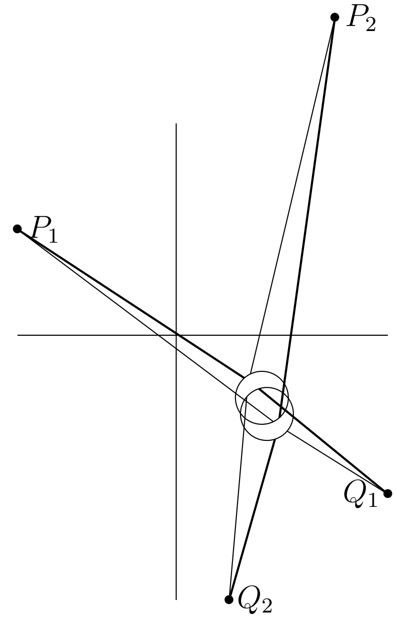
Figure 3.1. Example of geodesic.



Now we change the 8 in P_2 of the above example to 12, so that $\vec{P_1P_2}$ and $\vec{Q_1Q_2}$ are parallel in opposite directions; i.e., $h = (6, 4)$ and $k = (-3, -2)$.

The two equal paths are depicted in Figure 3.2. We have $w = \pm(2, -3)/\sqrt{13}$ in part (c) of Theorem 1.1. Also, $x = (3.2385, -2.3633)$ and $y = (3.4291, -2.9730)$ in (1.3) and $\beta = .4202$ in Corollary 2.5. The length of each of these paths in $F_0(\mathbb{R}^n, 2)$ is 28.375, compared with 28.213 for the straight-line path from P to Q, which is not in $F_0(\mathbb{R}^n, 2)$.

Figure 3.2. Example of two geodesics.



4. Unordered configuration space

As we stated in Theorem 1.7, the unordered configuration space $C(\mathbb{R}^n, 2)$ has a natural, Euclidean-related metric, and it is geodesically complete.

Proof of Theorem 1.7. We show that d_U satisfies the triangle inequality. Without loss of generality, assume

$$d((a,a'),(b,b')) \le d((a,a'),(b',b)) \quad \text{and} \quad d((b,b'),(c,c')) \le d((b,b'),(c',c)).$$

Then

$$\begin{aligned} d_U(\{a,a'\},\{c,c'\}) &\le d((a,a'),(c,c')) \\ &\le d((a,a'),(b,b')) + d((b,b'),(c,c')) \\ &= d(\{a,a'\},\{b,b'\}) + d(\{b,b'\},\{c,c'\}). \end{aligned}$$

The quotient topology on $C(\mathbb{R}^n, 2)$ comes from $\mathbb{R}^n \times (\mathbb{R}^n - \{0\})/\sim$ under $\{a,a'\} \mapsto \left(\frac{a+a'}{2}, \left[\frac{a-a'}{2}\right]\right)$. Our metric d_U corresponds to the metric $d([x],[x']) = \min(d(x,x'), d(x,-x'))$ on $(\mathbb{R}^n - \{0\})/\sim$, which gives the quotient topology.

If $d_U(\{a,a'\},\{b,b'\}) = d((a,a'),(b,b'))$, then the linear path $(1-t)(a,a') + t(b,b')$ lies in $F(\mathbb{R}^n, 2)$, so its equivalence class is in $C(\mathbb{R}^n, 2)$. [[As observed at the beginning of the proof of Theorem 1.1, the only thing that would prevent the path from being in $F(\mathbb{R}^n, 2)$ is if $b' - b$ is a negative multiple of $a' - a$. If this is the case, then

$$\begin{aligned} & d((a,a'),(b,b'))^2 - d((a,a'),(b',b))^2 \\ = \; & \|a-b\|^2 + \|a'-b'\|^2 - \|a-b'\|^2 - \|a'-b\|^2 \\ = \; & -2a\cdot b - 2a'\cdot b' + 2a\cdot b' + 2a'\cdot b \\ = \; & 2(a'-a)\cdot(b-b') \qquad (4.1) \\ > \; & 0, \end{aligned}$$

contradicting the assumption that $d_U(\{a,a'\},\{b,b'\}) = d((a,a'),(b,b'))$.]] ■

Remark 4.2. The analogue of Theorem 1.7 is valid for $C(\mathbb{R}^n, k)$ for any n and k.

Proposition 4.3. *For a, a', b, and b' in $\mathbb{R}^n$, $d((a,a'),(b,b')) = d((a,a'),(b',b))$ iff $(a'-a)\cdot(b'-b) = 0$. Let*

$$E_0 = \{(\{a,a'\},\{b,b'\}) \in C(\mathbb{R}^n,2)\times C(\mathbb{R}^n,2) : (a'-a)\cdot(b'-b) \neq 0\}.$$

There is a continuous geodesic motion-planning rule on E_0.

Proof. The first part follows as in (4.1). At each point of E_0, there is a unique choice of linear geodesic whose length equals $d_U(\{a,a'\},\{b,b'\})$, varying continuously with the point of E_0. ■

Now we can prove Theorem 1.8 about $\mathrm{GC}(C(\mathbb{R}^n,2))$.

Proof of Theorem 1.8. Let $E_1 = C(\mathbb{R}^n,2)\times C(\mathbb{R}^n,2) - E_0$, with E_0 as in Proposition 4.3. We need to describe subsets of E_1 on which we can make a continuous choice of whether to go from (a,a') to (b,b'), or to (b',b). Let $t_n = \mathrm{TC}(RP^{n-1})$. One motion-planning rule for RP^{n-1} is on the domain D_0 consisting of all pairs (ℓ_0,ℓ_1) such that $\ell_0\cdot\ell_1 \neq 0$. On D_0, rotate from ℓ_0 to ℓ_1 in their plane through the smaller arc ($< \pi/2$). Suppose D_i is one of the other t_n subsets of $RP^{n-1}\times RP^{n-1}$ on which there is a motion-planning rule s_i. For $(\ell_0,\ell_1) \in D_i$, s_i associates to an orientation (i.e., direction) on ℓ_0 an orientation on ℓ_1 by using the path from ℓ_0 to ℓ_1 specified by s_i. This association of orientations is continuous on D_i.

Let $E_i \subset E_1$ denote all $(\{a,a'\},\{b,b'\})$ such that, if ℓ_0 is the line through a and a', translated to pass through $\overline{0}$, and ℓ_1 the translated line passing through b and b', then $(\ell_0,\ell_1) \in D_i$. If $a > a'$ under the orientation of ℓ_0, choose the linear geodesic from (a,a') to (b,b'), where $b > b'$ under the associated orientation of ℓ_1. This is a continuous choice on E_i.

Thus we have partitioned $C(\mathbb{R}^n,2)\times C(\mathbb{R}^n,2)$ into t_n+1 subsets on which we have geodesic motion planning rules, so $\mathrm{GC}(C(\mathbb{R}^n,2)) \le \mathrm{TC}(RP^{n-1})$. Since, from (2.13), $C(\mathbb{R}^n,2)$ has the homotopy type of RP^{n-1}, we have the following string of inequalities, which imply the claimed equality.

$$\mathrm{GC}(C(\mathbb{R}^n,2)) \ge \mathrm{TC}(C(\mathbb{R}^n,2)) = \mathrm{TC}(RP^{n-1}) \ge \mathrm{GC}(C(\mathbb{R}^n,2)).$$

■

Note that our argument did not use that the motion-planning rules of [4] can be chosen to be geodesics.

5. A different metric

There is an obvious homeomorphism $F(\mathbb{R}^n,2) \to \mathbb{R}^n \times S^{n-1} \times \mathbb{R}^+$, where $\mathbb{R}^+ = (0,\infty)$, given by

$$(a,a') \mapsto \left(\frac{a'+a}{2}, \frac{a'-a}{\|a'-a\|}, \frac{\|a'-a\|}{2}\right).$$

In the notation of Theorem 1.1, with $\widehat{h} = h/\|h\|$, it is $(a,a') \mapsto (A,\widehat{h},\|h\|)$. Here A is the midpoint, and h the directed segment from the midpoint to the second point. The inverse sends (A,u,r) back to $(A-ru, A+ru)$.

We use the Euclidean metric on $\mathbb{R}^n$ and $\mathbb{R}^+$, and arclength metric d_S on S^{n-1}, and the product metric on their product to obtain a metric d' on $F(\mathbb{R}^n,2)$ which is geodesically complete. The formula, with B and k also as in Theorem 1.1, is

$$d'((a,a'),(b,b')) = \sqrt{\|B-A\|^2 + d_S(\widehat{h},\widehat{k})^2 + (\|k\|-\|h\|)^2}.$$

The unique geodesic from (a,a') to (b,b'), if $d_S(\widehat{h},\widehat{k}) < \pi$, is

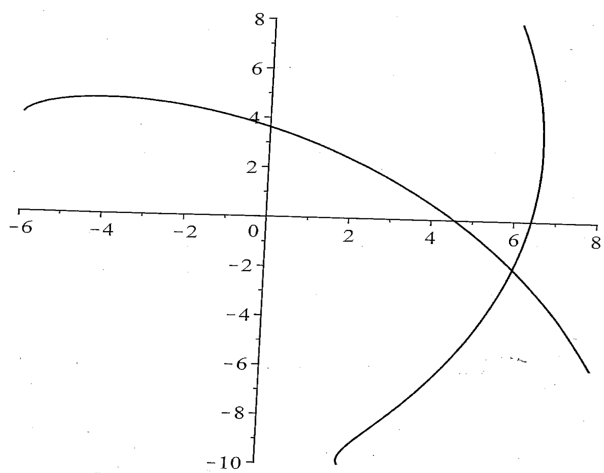
$$t \mapsto ((1-t)A+tB-((1-t)\|h\|+t\|k\|)u(t), (1-t)A+tB+((1-t)\|h\|+t\|k\|)u(t)),$$

where, similarly to Proposition 1.4, with $\alpha = d_S(\widehat{h}, \widehat{k})$,

$$u(t) = \frac{\sin((1-t)\alpha)\widehat{h} + \sin(t\alpha)\widehat{k}}{\sin\alpha}.$$

If $d_S(\widehat{h}, \widehat{k}) = \pi$, we use vector fields on S^{n-1} or $S^{n-1} - \{x_0\}$ to choose geodesics, and obtain an analogue of Theorem 1.6 for $F(\mathbb{R}^n, 2)$ in this metric. Figure 5.1 shows the path obtained in this way between the points that we used in the first example of Section 3.

Figure 5.1. Geodesic in $F(\mathbb{R}^2, 2)$ using "different" metric



There is an analogous metric on $C(\mathbb{R}^n, 2)$, but we will not discuss it here because the Euclidean-related metric considered earlier was already geodesically complete and highly satisfactory.

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