IMPUTING SPEED RECORDS

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A fellow long-distance runner and I recently compiled records for Pennsylvania residents of each age and sex at 50 km, 50 miles, 100 km, 100 miles, and 24 hours. ([1]) The 50 km distance does not have as many high-quality performances as 50 miles or 100 km because it is not generally considered to be a high-profile event. People who might be good at 50 km (31.07 miles) are more likely to put their best effort into the more popular marathon (26.2 miles) event.

There were several cases where the record for 100 km was less than 2 times the record for 50 km. In such a case, one can easily deduce that the person broke the 50 km record during either the first half or the second half of the 100 km race.[1] But how about 50 miles? How fast must one run 50 miles in order to assert that they ran some 50 km subinterval faster than some prescribed time?

One approach would be to estimate that your time for 50 km must be at least as good as your 50-mile time minus the world record for 18.93 miles (50 miles minus 50 km). My friend rejected that approach, in part because the world record might change. Removing the assumption that the speed during a certain portion of the race has some fixed bound leads a mathematician into the world of abstraction. Allow the possibility that distance covered is any continuous nondecreasing function of elapsed time. We find that, given a person's time for 50 miles, in order to know that during some 50 km segment of the race they broke the 50 km record, it must be the case that they ran 50 miles faster than the 50 km record. The purpose of this note is to prove this result and a generalization.

Definition 0.1.

- *Let* $\mathcal{F} = \{f : [0,1] \to [0,1] :\ f(0) = 0,\ f(1) = 1$*, and* f *is continuous and nondecreasing*$\}$.

[1]For official purposes, such a record established during the second half of a race would not count as a record.

- *If* $f \in \mathcal{F}$ *and* $0 < D \leq 1$, *let*
$$m(f,D) = \min\{t_1 - t_0 : \ f(t_1) - f(t_0) \geq D\}.$$
- *If* $0 < D \leq 1$, *let* $M(D) = \sup\{m(f,D) : \ f \in \mathcal{F}\}$.

Here, for simplicity, we have normalized the distance and time to 1 unit each. We interpret $f(t)$ to be the distance covered up to time t. Then $m(f,D)$ is the person's fastest time for a subinterval of distance D. (A compactness argument implies that the "inf" that one would ordinarily write in a definition such as that of $m(f,D)$ can actually be written as "min;" i.e. the minimum is achieved.) Then $M(D)$ is the best that you can necessarily conclude about someone's best time for distance D, given that they covered the unit distance in unit time. Our theorem is as follows.

Theorem 0.2. *If* n *is any positive integer and* $\frac{1}{n+1} < D \leq \frac{1}{n}$, *then* $M(D) = 1/n$.

Rescaling, we obtain the following interpretation. If a person runs distance L in time T, and $\frac{1}{n+1} < L_0/L \leq \frac{1}{n}$, then he certainly ran some interval of length L_0 in time less than or equal to T/n, and this is the best that can be concluded. In particular, since $\frac{1}{2} < 31.07/50$, the best that we can conclude is that a 50-mile runner ran some 50 km segment at least as fast as his 50-mile time.

The discontinuous nature of $M(D)$ seems somewhat counterintuitive. For example, it says that if we know that someone ran a mile in exactly 4 minutes, then we can conclude that he ran some quarter mile in 1 minute or less, but for a distance slightly greater than a quarter mile, such as .2500001 miles, all we can conclude is that he ran it in 4/3 minutes (1:20) or less. After I had completed this article, my runner-friend (a nonmathematician) pointed out to me an article ([2]) in a running magazine written by the mathematician/runner Stan Wagon, reaching the same conclusions in a less technical fashion.

The proof will utilize the following proposition, which is immediate from the definitions.

Proposition 0.3. *For any* $f \in \mathcal{F}$, $m(f,D)$ *is a nondecreasing function of* d. *Consequently,* $M(D)$ *is a nondecreasing function of* D.

Proof of Theorem 0.2. For any positive integer n and any sufficiently small positive number ϵ, define a function $f_{n,\epsilon}$ in $\mathcal{F}$ by

$$f_{n,\epsilon}(t) = \begin{cases} \frac{a}{n+1} & \text{if } \frac{a-1}{n} + \epsilon \le t \le \frac{a}{n}, \quad 1 \le a < n \\ \frac{n}{n+1} & \text{if } \frac{n-1}{n} + \epsilon \le t \le 1 - \epsilon \\ \frac{a+s}{n+1} & \text{if } t = \frac{a}{n} + s\epsilon, \quad 0 \le a < n, \quad 0 \le s \le 1 \\ 1 - \frac{s}{n+1} & \text{if } t = 1 - s\epsilon, \quad 0 \le s \le 1. \end{cases}$$

Thus $f_{n,\epsilon}$ is a continuous approximation of a step function with steps of height $1/(n+1)$ and width $1/n$. The approximation is linear within distance ϵ of each step point.

Let $D > \frac{1}{n+1}$. If $\epsilon < \frac{1}{2n}$, then $m(f_{n,\epsilon}, D) \ge \frac{1}{n} - \epsilon$. To see this, let

$$\Delta = \begin{cases} \frac{1}{n} & \text{if } t_1 \le 1 - \epsilon \\ \frac{1}{n} - \epsilon & \text{if } t_1 > 1 - \epsilon. \end{cases}$$

Then $f_{n,\epsilon}(t_1) - f_{n,\epsilon}(t_1 - \Delta) = \frac{1}{n+1}$, from which we deduce that $t_1 - t_0 \le \frac{1}{n} - \epsilon$ implies

$$f_{n,\epsilon}(t_1) - f_{n,\epsilon}(t_0) \le \frac{1}{n+1} < D,$$

and hence $m(f_{n,\epsilon}, D) \ge \frac{1}{n} - \epsilon$. Thus $M(D) \ge \frac{1}{n} - \epsilon$ for all sufficiently small positive ϵ, and hence

$$M(D) \ge 1/n. \tag{0.4}$$

On the other hand, for any $f \in \mathcal{F}$,

$$1 = \sum_{i=1}^{n} (f(\tfrac{i}{n}) - f(\tfrac{i-1}{n})).$$

Thus, for some i, $f(\frac{i}{n}) - f(\frac{i-1}{n}) \ge \frac{1}{n}$. Thus $m(f, \frac{1}{n}), \le \frac{1}{n}$, and hence

$$M(\tfrac{1}{n}) \le \tfrac{1}{n}. \tag{0.5}$$

Our desired conclusion now follows from the proposition plus (0.4) and (0.5). Indeed,

$$\tfrac{1}{n} \le M(D) \le M(\tfrac{1}{n}) \le \tfrac{1}{n}.$$

□

References

[1] D. Davis and N. Marshall, *Ultramarathon records for Pennsylvania residents by age and sex*, http://www.lehigh.edu/∼dmd1/parecords.

[2] S. Wagon, *"But did he really. . . "*, UltraRunning (1983).

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