REAL PROJECTIVE SPACE AS A SPACE OF PLANAR POLYGONS

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Abstract. We describe an explicit homeomorphism between real projective space RP^{n-3} and the space $\overline{M}_{n,n-2}$ of all isometry classes of n-gons in the plane with one side of length $n-2$ and all other sides of length 1. This makes the topological complexity of real projective space more relevant to robotics.

1. Introduction

The topological complexity, $\mathrm{TC}(X)$, of a topological space X is, roughly, the number of rules required to specify how to move between any two points of X.([4]) This is relevant to robotics if X is the space of all configurations of a robot.

A celebrated theorem in the subject states that, for real projective space RP^n with $n \neq 1$, 3, or 7, $\mathrm{TC}(RP^n)$ is 1 greater than the dimension of the smallest Euclidean space in which RP^n can be immersed.([5]) This is of interest to algebraic topologists because of the huge amount of work that has been invested during the past 60 years in studying this immersion question. See, e.g., [6], [9], [1], and [2]. In the popular article [3], this theorem was highlighted as an unexpected application of algebraic topology.

But, from the definition of RP^n, all that $\mathrm{TC}(RP^n)$ really tells is how hard it is to move efficiently between lines through the origin in $\mathbb{R}^{n+1}$, which is probably not very useful for robotics. Here we show explicitly how RP^n may be interpreted to be the space of all polygons of a certain type in the plane. The edges of polygons can be thought of as linked arms of a robot, and so $\mathrm{TC}(RP^n)$ can be interpreted as telling how many rules are required to tell such a robot how to move from any configuration to any other.

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Let $M_{n,r}$ denote the moduli space of all oriented n-gons in the plane with one side of length r and the rest of length 1, where two such polygons are identified if one can be obtained from the other by an orientation-preserving isometry of the plane. These n-gons allow sides to intersect. Since any such n-gon can be uniquely rotated so that its r-edge is oriented in the negative x-direction, we can fix vertices $\mathbf{x}_0 = (0,0)$ and $\mathbf{x}_{n-1} = (r,0)$ and define

$$M_{n,r} = \{(\mathbf{x}_1, \ldots, \mathbf{x}_{n-2}) : d(\mathbf{x}_{i-1}, \mathbf{x}_i) = 1,\ 1 \le i \le n-1\}. \tag{1.1}$$

Here d denotes distance between points in the plane.

Most of our work is devoted to proving the following theorem.

Theorem 1.2. *If $n-2 \le r < n-1$, then there is a $\mathbb{Z}/2$-equivariant homeomorphism $\Phi : M_{n,r} \to S^{n-3}$, where the involutions are reflection across the x-axis in $M_{n,r}$, and the antipodal action in the sphere.*

Taking the quotient of our homeomorphism by the $\mathbb{Z}/2$-action yields our main result. It deals with the space $\overline{M}_{n,r}$ of isometry classes of planar $(1^{n-1}, r)$-polygons. This could be defined as the quotient of (1.1) modulo reflection across the x-axis.

Corollary 1.3. *If $n-2 \le r < n-1$, then $\overline{M}_{n,r}$ is homeomorphic to RP^{n-3}.*

These results are not new. It was pointed out to the author after preparation of this manuscript that the result is explicitly stated in [8, Example 6.5], and proved there, adapting an argument given much earlier in [7]. The result of our Corollary 1.3 was also stated as "well known" in [10]. Nevertheless, we feel that our explicit, elementary homeomorphism may be of some interest.

2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. Let J^m denote the m-fold Cartesian product of the interval $[-1,1]$, and $S^0 = \{\pm 1\}$. Our model for S^{n-3} is the quotient of $J^{n-3} \times S^0$ by the relation that if any component of J^{n-3} is ± 1, then all subsequent coordinates are irrelevant. That is, if $t_i = \pm 1$, then

$$(t_1, \ldots, t_i, t_{i+1}, \ldots, t_{n-2}) \sim (t_1, \ldots, t_i, t'_{i+1}, \ldots, t'_{n-2}) \tag{2.1}$$

for any $t'_{i+1}, \ldots, t'_{n-2}$. This is just the iterated unreduced suspension of S^0, and the antipodal map is negation in all coordinates. An explicit homeomorphism of this model with the standard S^{n-3} is given by

$$(t_1, \ldots, t_{n-2}) \leftrightarrow (x_1, \ldots, x_{n-2}),$$

with

$$x_i = t_i \prod_{j=1}^{i-1} \sqrt{1 - t_j^2}, \qquad t_i = \frac{x_i}{\sqrt{1 - x_1^2 - \cdots - x_{i-1}^2}} \text{ if } \sum_{j=1}^{i-1} x_i^2 < 1.$$

Then $t_i = \pm 1$ for the smallest i for which $x_1^2 + \cdots + x_i^2 = 1$.

Let $\mathcal{P} \in M_{n,r}$ be a polygon with vertices $\mathbf{x}_i$ as in (1.1). We will define the coordinates $t_i = \phi_i(\mathcal{P})$ of $\Phi(\mathcal{P})$ under the homeomorphism Φ of Theorem 1.2.

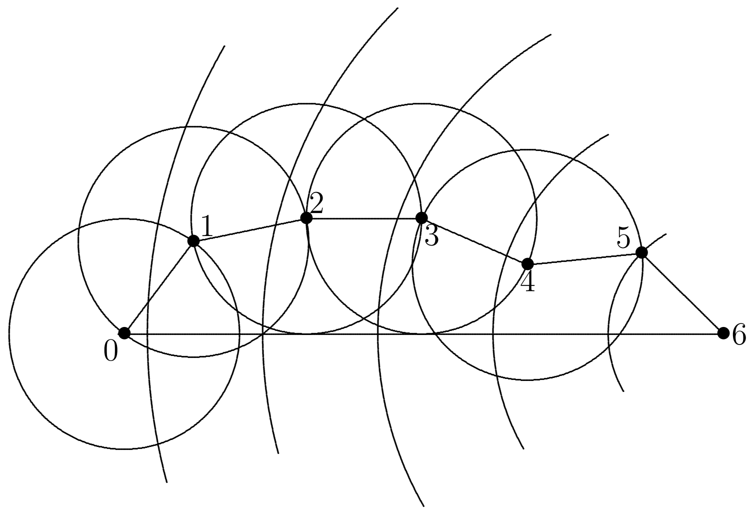
For $0 \le i \le n-2$, we have

$$n - 2 - i \le d(\mathbf{x}_i, \mathbf{x}_{n-1}) \le n - 1 - i. \tag{2.2}$$

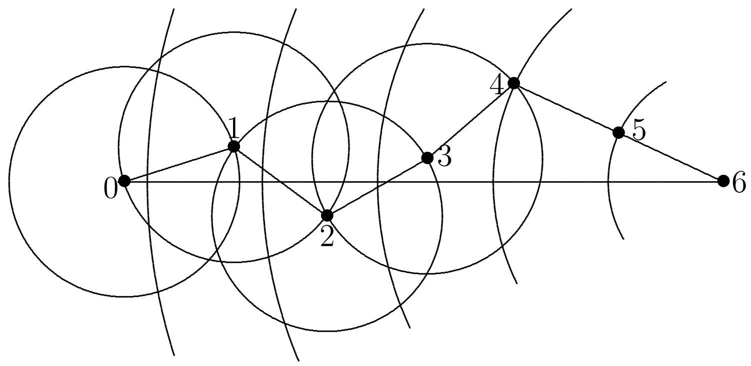
The first inequality follows by induction on i from the triangle inequality and its validity when $i = 0$. The second inequality also uses the triangle inequality together with the fact that you can get from $\mathbf{x}_i$ to $\mathbf{x}_{n-1}$ by $n-1-i$ unit segments. The second inequality is strict if $i = 0$ and is equality if $i = n-2$. Let i_0 be the minimum value of i such that equality holds in this second inequality. Then the vertices $\mathbf{x}_{i_0} \ldots, \mathbf{x}_{n-1}$ must lie evenly spaced along a straight line segment.

Let $C(\mathbf{x}, t)$ denote the circle of radius t centered at $\mathbf{x}$. The inequalities (2.2) imply that, for $1 \le i \le i_0$, $C(\mathbf{x}_{n-1}, n-1-i)$ cuts off an arc of $C(\mathbf{x}_{i-1}, 1)$, consisting of points $\mathbf{x}$ on $C(\mathbf{x}_{i-1}, 1)$ for which $d(\mathbf{x}, \mathbf{x}_{n-1}) \le n-1-i$. Parametrize this arc linearly, using parameter values -1 to 1 moving counterclockwise. The vertex $\mathbf{x}_i$ lies on this arc. Set $\phi_i(\mathcal{P})$ equal to the parameter value of $\mathbf{x}_i$. If $i = i_0$, then $\phi_i(\mathcal{P}) = \pm 1$, and conversely.

The following diagram illustrates a polygon with $n = 7$, $r = 5.2$, and $i_0 = 5$. We have denoted the vertices by their subscripts. The circles from left to right are $C(\mathbf{x}_i, 1)$ for i from 0 to 4. The arcs centered at $\mathbf{x}_6$ have radius 1 to 5 from right to left. We have, roughly, $\Phi(\mathcal{P}) = (.7, .6, .5, -.05, 1)$.



Here is another example, illustrating how the edges of the polygon can intersect one another, and a case with $i_0 < n-2$. Again we have $n = 7$ and $r = 5.2$. This time, roughly, $\Phi(\mathcal{P}) = (.2, -.4, .4, 1, t_5)$, with t_5 irrelevant. Because $i_0 = 4$, we did not draw the circle $C(\mathbf{x}_4, 1)$.



That Φ is well defined follows from (2.1); once we have $t_i = \pm 1$, which happens first when $i = i_0$, subsequent vertices are determined and the values of subsequent t_j are irrelevant. Continuity follows from the fact that the unit circles vary continuously with the various $\mathbf{x}_i$, hence so do the parameter values along the arcs cut off. Bijectivity follows from the construction; every set of t_i's up to the first ± 1 corresponds to a unique polygon, and ± 1 will always occur. Since it maps from a compact space to a Hausdorff space, Φ is then a homeomorphism. Equivariance with respect to the involution is also clear. If you flip the polygon, you flip the whole picture, including the unit circles, and this just negates all the t_i's.

We elaborate slightly on the surjectivity of Φ. The arc on $C(\mathbf{x}_0, 1)$ cut off by $C(\mathbf{x}_{n-1}, n-2)$ is determined by n and r. Given a value of t_1 in $[-1, 1]$, the vertex $\mathbf{x}_1$ is now determined on this arc. Now the arc on $C(\mathbf{x}_1, 1)$ cut off by $C(x_{n-1}, n-3)$ is determined, and a specified value of t_2 determines the vertex $\mathbf{x}_2$. All subsequent vertices of an n-gon are determined in this manner.

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