Contemporary Mathematics
Volume 12, 1982

ON THE COHOMOLOGY OF MO<8>

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1. INTRODUCTION

Let BO<8> denote the classifying space for vector bundles trivial on the 7-skeleton, and MO<8> the associated Thom spectrum. In this paper some progress is made toward the determination of the structure of the A-module H*MO<8>. Here A denotes the mod 2 Steenrod algebra and all cohomology groups have Z_2-coefficients. Let A_r denote the subalgebra of A generated by $\{Sq^j : j \leq 2^r\}$. Our main result, Theorem 2.3, is the structure of H*MO<8> as an A_1-module. This result suggests conjectures about $\mathrm{Ext}_A(H^*(P_{-\infty}^{\infty}\wedge MO\langle 8\rangle), Z_2)$ and indeed about the spectrum $P_{-\infty}^{\infty}\wedge MO\langle 8\rangle$, where $P_{-\infty}^{\infty}$ is the inverse limit spectrum studied in [Lin].

The goal of the program initiated here is a result similar to the result of [ABP]—

$H^*MSpin \approx \bigoplus_J A/A\bar{A}_1 \cdot x_J \oplus \bigoplus_K A/ASq^3 \cdot y_K \oplus$ free A-module. Since the annihilator of the Thom class of H*MO<8> is A_2, we would expect H*MO<8> to be an extended A_2-module; i.e., $A \otimes_{A_2} N$ for some A_2-module N, but there do not seem to be any general results which guarantee this. Margolis ([Mar 21.4]) proves a filtered version—that such a comodule has a filtration with R_i/R_{i-1} an extended A_2-module with generators of degree i. The splitting of H*MSpin was induced by a splitting of spectra—$MSpin \approx \bigvee \Sigma^{4|J|}bo \vee \bigvee \Sigma^{4|K|}bo^{<2>} \vee K$, where $bo^{<2>}$ is the spectrum obtained from bo (localized at 2) by killing classes of Adams filtration less than 2 and K is a wedge of $K(Z_2)$-spectra. Such a splitting for MO<8> is unlikely in light of the fact that the first summand of

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$H^*MO\langle 8\rangle$ is $A//A_2$ ([BM1]), and this cannot be realized as the cohomology of a spectrum ([DM2]).

To make matters worse, $H^*MO\langle 8\rangle$ cannot even be written as a direct sum of cyclic A-modules, which is somewhat surprising considering the nice splitting of H^*MSpin. This result (2.8) follows quite easily from 2.3.

The action of A on $H^*MO\langle 8\rangle$ can be written explicitly using the Thom isomorphism $H^*BO\langle 8\rangle \xrightarrow{\phi} H^*MO\langle 8\rangle$ and Stong's result ([S]) that under $H^*BSO \xrightarrow{p^*} H^*BO\langle 8\rangle$,

$$H^*BO\langle 8\rangle \approx Z_2[p^*w_n : \alpha(n-1) \geq 3].$$

The Wu relations give the A-action in H^*BSO, but this is complicated in $H^*BO\langle 8\rangle$ by the fact that for $\alpha(n-1) < 3$, p^*w_n may be nonzero. Indeed, by [BM2], if $\alpha(n-1) = 1$ or 2, then $p^*w_n = 0$ if $n \leq 33$, but p^*w_n is a nonzero decomposable if $n \geq 34$. For example,

$$p^*w_{34} = w_8w_{26} + w_{12}w_{22} + w_{14}w_{20}$$
$$p^*w_{49} = w_{20}w_{29} + w_{22}w_{27} + w_{23}w_{26},$$

where classes on the right hand side denote image under p^*. These relations can be explicitly determined by applying the Wu relations to $p^*(Sq^{2^i}Sq^{2^{i-1}}\cdots Sq^2Sq^1w_2) = 0$ and $p^*(Sq^{2^{i+j}+2^{j-1}}\cdots Sq^{2^{i+2}+2}Sq^{2^{i+1}+1}Sq^{2^i}\cdots Sq^2w_4) = 0$. The preceding examples are the cases $i = 4$, $j = 0$ and $i = 0$, $j = 4$. These relations become much more complicated for w_n with $n \geq 65$.

Using a PASCAL program to calculate the A-action on $H^*MO\langle 8\rangle$, we have showed that through degree 51 $H^*MO\langle 8\rangle$ splits as

$$(\Sigma^0 + \Sigma^{16} + 2\Sigma^{32} + 4\Sigma^{48})A//A_2 \oplus (\Sigma^{20} + 2\Sigma^{36})A/A(Sq^1, Sq^5, Sq^6, Sq^{13})$$
$$\oplus\, \Sigma^{40}A/A(Sq^1, Sq^9) \oplus A(g_{44}, g_{49})/(Sq^1g_{44}, Sq^5g_{44}, Sq^1g_{49}, Sq^2g_{49} + Sq^7g_{44}).$$

In [G1] Giambalvo claimed to have written the splitting through degree 49, but our results begin to differ with his in 46.

In a second paper ([G2]) Giambalvo claimed that a relation in degree 55 implied that $A//A_2 \cdot U$ did not split, which was shown to be false in [BM1]. This mistake was due to a misunderstanding in the concept of splitting, and not to an incorrect calculation. It is conceivable that our result on the A_1-splitting of $H^*MO\langle 8\rangle$ might enable us to guess the structure of $H^*MO\langle 8\rangle$ as an extended A_2-module, and then to use the method of [ABP] to prove the desired splitting result.

Using the techniques of [DM1], such a splitting of $H^*MO\langle 8\rangle$ would allow a calculation of $Ext_A(H^*MO\langle 8\rangle, Z_2)$, the E_2-term of the Adams spectral sequence converging to the <8>-cobordism ring. Another possible application of this knowledge is to obstruction theory as discussed in [DM1], using the MO<8>-orientablity of vector bundles trivial on the 7-skeleton ([DGIM]).

2. THE A_1-STRUCTURE OF $H^*MO\langle 8\rangle$

We begin by reviewing some A_1-modules and establishing notation. We work in the category of bounded below stable A_1-modules of finite type, which means that we ignore free A_1-modules. Adams ([AP]) showed that the only invertible modules are $\Sigma^k I^\ell J^\varepsilon$, where $k, \ell \in Z$ and $\varepsilon \in Z_2$, Σ^k is Z_2 in degree k, $I = \ker(A_1 \longrightarrow Z_2)$, $I^{-\ell}$ is the dual of $I^\ell = I \otimes \ldots \otimes I$, and $J = \Sigma^{-2}A_1/A_1Sq^3$. These modules for $\ell \geq 0$ are (stably) the same as the $Q_{j,n}$ considered in [DM3] and [DGM] and T, S, Y and Z of [Mi]. The $Q_{j,n}$ and T-Z have the advantage of being minimal (no free A_1's) so that one has a better picture of the module. Adams' modules have the advantage of nicer multiplicative properties and nicer formulas for our work. For $j \in Z_4$ and $n \geq 0$, $Q_{j,n}$ defined inductively to be the nontrivial extension of A_1-modules.

$$0 \longrightarrow A/ASq^1 \longrightarrow Q_{j,n} \longrightarrow \Sigma^4 Q_{j,n-1} \longrightarrow 0$$

with $Q_{0,0} = Z_2$, $Q_{1,0} = A_1/(Sq^1, Sq^2Sq^3)$, $Q_{2,0}$ Adams' J, and $Q_{3,0} = \Sigma^{-3}A_1/(Sq^2)$. Conversion between the two notations is provided by

PROPOSITION 2.1. For $0 \le \Delta \le 3$, $I^{4a+\Delta} \approx \Sigma^{4a+\Delta}Q_{-\Delta,2a+1-\delta}$ and $I^{4a+\Delta}J \approx \Sigma^{4a+\Delta}Q_{2-\Delta,2a-1+\Delta+\delta}$, where $\delta = \delta_{0,\Delta}$ is the Kronecker delta.

In particular we will need the stable equivalences

$$Q_{1,0} \approx \Sigma^{-1}IJ, \quad Q_{1,2\ell+1} \approx (\Sigma^{-1}I)^{4\ell+3}.$$

Ext charts for Adams' modules have a nice form in filtration > 0.

PROPOSITION 2.2. $\mathrm{Ext}_{A_1}^{s,t}(\Sigma^k I^\ell, Z_2) \approx \mathrm{Ext}_{A_1}^{s+\ell,t-k}(Z_2, Z_2)$ if $s > 0$;

$\mathrm{Ext}_{A_1}^{s,t}(\Sigma^k I^\ell J, Z_2) \approx \mathrm{Ext}_{A_1}^{s+\ell+2,t-k+6}(Z_2, Z_2)$ if $s > 0$ and $t-s \ge k+\ell$.

This proposition could be restated by:

$$A \otimes_{A_1} I^\ell \approx H^*(\Sigma^\ell bo^{\langle\ell\rangle}) \quad \text{and} \quad A \otimes_{A_1} I^\ell J \approx H^*(\Sigma^\ell bsp^{\langle\ell-1\rangle}).$$

We can now state our main theorem.

THEOREM 2.3. As a stable A_1-module $H^*MO\langle 8\rangle$ is isomorphic to

$$\bigoplus_{S,T} \Sigma^{4|S|+4|T|} J^{|S|+|T|} (\Sigma^{-1}I)^{\Sigma_{t\in T}(2^{\nu(t)+1}-1)}$$

where

S ∈ {nondecreasing sequences of integers $s \ge 2$
such that $s-1$ is not an even 2-power}

T ∈ {increasing sequences of integers t with $\alpha(t) = 2$},

and $|S|$ is the sum of the elements of S. This module will be illustrated at the end of this section.

A key lemma is

LEMMA 2.4. As an A_1-module $H^*BO\langle 8\rangle \approx Z_2[g_n : \alpha(n-1) \ge 3]$, where for $j = 1$ and 2

$$Sq^j g_n = \begin{cases} \binom{n-1}{j} g_{n+j} & \text{if } \alpha(n+j-1) \ge 3 \\ 0 & \text{otherwise} \end{cases}.$$

The analogue of this lemma for A_2 fails: There is no generator in degree 40 annihilated by Sq^1, Sq^4Sq^2, and $Sq^2Sq^3Sq^4$. This is ultimately due to the fact that p^*w_{49} is nonzero. The failure of this lemma for A_2 is a major barrier for the calculation of the splitting of $H^*MO\langle 8\rangle$.

PROOF OF LEMMA. We use the A-action on $H^*BO\langle 8\rangle$ as in [ABP] or [BDP]. If $R = (r_1, r_2, \ldots)$ is a finite sequence of nonnegative integers, let $d(R) = \Sigma(2^i-1)r_i$ and $d'(R) = \Sigma 2^{i-1}r_i$. Let

$$R_n = \begin{cases} \{2^i\}, & \text{if } n = 2^i \geq 8 \\ \{R : d(R) = n,\ d'(R) = 2^i,\ r_1 = 2^i - 2^{j+1},\ r_2 \equiv 0(4),\ r_3 \equiv 0(2)\} & \\ \qquad \text{if } 2^i + 2^j \leq n \leq 2^i + 2^{j+1}\ {}_{i>j} & \end{cases}$$

and let $g_n = \Sigma_{R\in R_n}(1)Sq(R)$. The lemma follows easily from the following statements:

(i) $g_n = 0$ if $\alpha(n-1) < 3$

(ii) $Sq^1 g_{2n} = g_{2n+1}$

(iii) $Sq^2 g_{4n-\varepsilon} = g_{4n+2-\varepsilon}$ for $\varepsilon \in \{0,1\}$

(iv) $g_n \equiv p^*w_n$ mod dec if $\alpha(n) \leq 2$, $\alpha(n) + \nu(n) \geq 4$.

Indeed, other $Sq^j g_n$ follow from (ii), (iii) and Adams' relations, while (iv) for other values of n follows from Wu relations. [Suppose we proved (iv) for g_{4n}. For $d = 1, 2, 3,$ or 5, let $S^d = Sq^1$, Sq^2, Sq^1Sq^2, or $Sq^2Sq^1Sq^2$, respectively. If g_{4n+d} is nonzero, then it is indecomposable, since $S^d w_{4n} \equiv w_{4n+d}$ mod dec. If $\alpha(4n+4) > 2$, then $Sq^1 g_{4n+4} = g_{4n+5}$ is nonzero and is indecomposable by induction $(S^5 g_{4n})$. Since $Sq^1(\text{dec}) = \text{dec}$, this implies g_{4n+4} is indecomposable, extending the induction.]

(i) is proved by using:

(1) $2^i = 2^i - 2^{j+1} + 2r_2 + 4r_3 + \ldots$

(2) $2^i + 2^j + 1 = 2^i - 2^{j+1} + 3r_2 + 7r_3 + \ldots$

to show R_n is empty. Indeed, $2(2) - 3(1)$ is incompatible with $r_3 \equiv 0(2)$. (iv) follows from [BDP, 3.1(c)] since $R_{2^i+2^j} = \{(2^i - 2^{j+1}, 2^j)\}$.

We prove (iii); (ii) is similar but easier. (iii) is equivalent to $Sq^2\Sigma_{R_{4n-\varepsilon}}\chi Sq(R)U = \Sigma_{R_{4n+2-\varepsilon}}\chi Sq(R)U$ in $H^*MO\langle 8\rangle$, which requires $Sq^2\Sigma_{R_{4n-\varepsilon}}\chi Sq(R) = \Sigma_{R_{4n+2-\varepsilon}}\chi Sq(r)$ in $A//A_2$, or equivalently

$$\sum_{R_{4n-\varepsilon}} Sq(R)\cdot Sq^2 \equiv \sum_{R_{4n+2-\varepsilon}} Sq(R) \bmod \bar{A}_2A. \qquad (3)$$

Since $\Sigma_{d(R)=m}\, Sq(R) = \chi Sq^m$, applying χ to the Adem relation for $Sq^2Sq^{4n-\varepsilon}$ shows

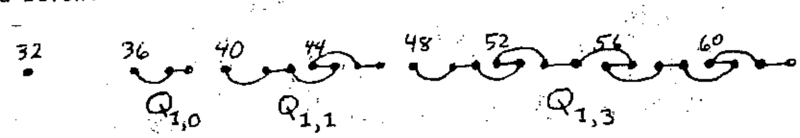
$$\sum_{d(R)=4n-\varepsilon} Sq(R)\cdot Sq^2 \equiv \sum_{d(R)=4n+2-\varepsilon} Sq(R) \bmod \bar{A}_2A. \qquad (4)$$

In $A/\bar{A}_2A$, $Sq(R)$ is nonzero iff $r_1 \equiv 0(8)$, $r_2 \equiv 0(4)$, and $r_3 \equiv 0(2)$, and so we may restrict (4) to such terms. Since RSq^2 cannot change r_1 from one multiple of 8 to another, (4) remains true if sums are restricted to a fixed multiple of 8; e.g., $2^i - 2^{j+1}$. If $Sq(T)$ occurs in the product $Sq(R)\cdot Sq^k$, then $d'(T) = d'(R)$ iff $t_{0,1} = 0$ in the Milnor matrix. This will be true if $k < 8$ and R and T begin with a multiple of 8. Thus (4) is true if sums are restricted to fixed $d'(R)$, establishing (3). ■

COROLLARY 2.5. $H^*BO\langle 8\rangle$ is a polynomial algebra on generators which as as A_1-module form a split summand

$$\bigoplus_{i\geq 3} \Sigma^{2^i} Z_2 \oplus \bigoplus_{\alpha(n)=2} \Sigma^{4n} Q_{1,2^{\nu(n)}-1}.$$

For example, between degrees 32 and 63 the A_1-action on the generators is as depicted below:



If M is an A_1-module, let $P(M)$ denote the A_1-module $\bigoplus_{k\geq 0} M^{\otimes k}/S_k$, where the symmetric group S_k acts by permutation of factors. Since $P(M \oplus N) \approx P(M) \otimes P(N)$, the following result is immediate from 2.5.

COROLLARY 2.6. As an A_1-module

$$H^*BO\langle 8\rangle \approx \bigotimes_{i\geq 3} P(\Sigma^{2^i} Z_2) \otimes \bigotimes_{\alpha(n)=2} P(\Sigma^{4n} Q_{1,2^{\nu(n)}-1}).$$

The last main ingredient is

LEMMA 2.7. $P(\Sigma^k Q_{1,\ell})$ is stably A_1-isomorphic to

$$(Z_2 \oplus \Sigma^k Q_{1,\ell}) \otimes \bigoplus_S \Sigma^{4|S|+2k\#(S)} J^{|S|}$$

where $S \in$ {nondecreasing sequences from $\{i : 0 \leq i \leq 2\ell + 1\}$} and $\#(S)$ is the number of elements in S.

Note that Adams' J satisfies $J^2 \approx Z_2$, so that $J^{|S|}$ in 2.7 depends only upon $|S|$ mod 2.

PROOF OF 2.7. Let $M = \Sigma^k Q_{1,\ell}$ and x_j the nonzero element of M of degree j (if one exists). Then

$$H_*(M; Q_0) \approx Z_2\left[\{x_{k+2j}^2 : 0 < j \leq 2\ell + 1\} \cup \{x_k\}\right]$$

$$H_*(M; Q_1) \approx Z_2\left[\{x_{k+2j}^2 : 0 \leq j < 2\ell + 1\} \cup \{x_{k+4\ell+2}\}\right].$$

[As a Q_0-module,

$$PM \approx P(\Sigma^k Z_2) \otimes \bigotimes_{j=1}^{2\ell+1} P\langle x_{k+2j}, Q_0 x_{k+2j}\rangle,$$

and the result follows from the Kunneth formula for $H_*(M; Q_0)$ and the fact that $H_*(P\langle x, Q_0 x\rangle; Q_0) \approx Z_2[x^2].$] Let

$$T_j = \begin{cases} Z_2 & j \text{ even} \\ J & j \text{ odd} \end{cases}.$$

For $0 \leq j \leq 2\ell + 1$ there are A_1-homomorphisms $\Sigma^{2k+4j} T_j \xrightarrow{\phi_j} PM$ defined by

$$\phi_j(\text{gen}) = \begin{cases} x_{k+2j}^2 + x_{k+2j-1}\, x_{k+2j+1} & j \text{ even} \\ x_{k+2j-2}\, x_{k+2j} & j \text{ odd} \end{cases}$$

Using the multiplication $PM \otimes PM \longrightarrow PM$ and the inclusion $M \xrightarrow{i} PM$, we get

$$(Z_2 \oplus M) \otimes \bigoplus_S \bigotimes_{j\in S} \Sigma^{2k+4j} T_j \xrightarrow{i\otimes\phi} PM$$

inducing an isomorphism in Q_0- and Q_1-homology, which implies it is a stable isomorphism by [AM].■

Now we combine 2.6 and 2.7, and use 2.1 to convert into Adams' notation, so that the mulitplication is easier. Note that we could not have switched to Adams' modules earlier because $P(M)$ and $P(M \oplus A_1)$ are not stably equivalent. This yields

$$\bigotimes_{i\geq 3} P(\Sigma^{2^i} Z_2) \otimes \bigotimes_{\alpha(n)=2} ((Z_2 \oplus \Sigma^{4n}(\Sigma^{-1}I)^{2^{\nu(n)+1}-1}J^n) \otimes \bigoplus_{K\in\mathcal{J}_n} \Sigma^{4|K|+8n\cdot\#(K)}J^{|K|}) \tag{5}$$

as stably A_1-equivalent to $H^*BO\langle 8\rangle$. Here $\mathcal{J}_n$ is the set of non-increasing sequences from $\{0, \ldots, 2^{\nu(n)+1}-1\}$. Let $\mathcal{J}'_n$ be the set of nonincreasing sequences from $\{2n, \ldots, 2n+2^{\nu(n)+1}-1\}$, and $\mathcal{P}$ the set of sequences of even 2-powers. Then (5) becomes

$$\bigoplus_{S_1\in\mathcal{P}} \Sigma^{4|S_1|} \otimes \bigotimes_T \Sigma^{4|T|}J^{|T|}(\Sigma^{-1}I)^{\Sigma(2^{\nu(n)+1}-1)} \otimes \bigotimes_{\alpha(n)=2} \bigoplus_{S_2\in\mathcal{J}'_n} \Sigma^{4|S_2|}J^{|S_2|}$$

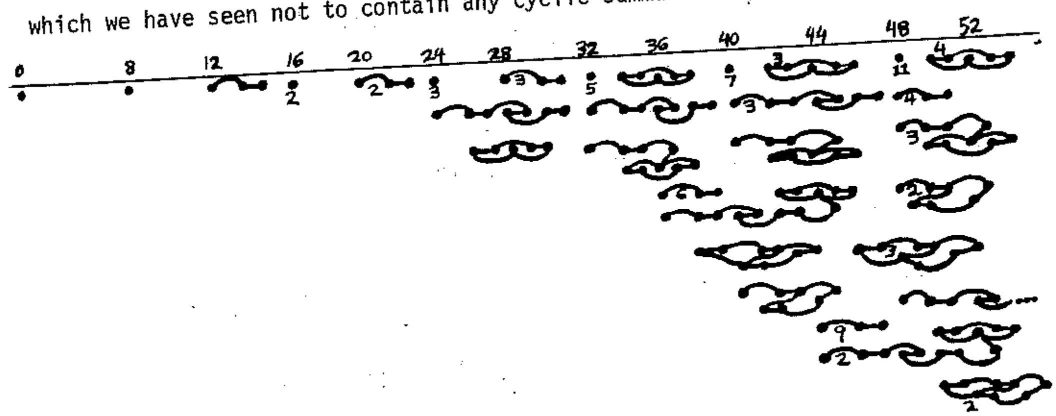
where T is as in 2.3. The sets from which the sequences in $\mathcal{P}$ and $\mathcal{J}'_n$ are drawn are disjoint, so that these sequences can be combined, yielding 2.3 as the stable A_1-structure of $H^*BO\langle 8\rangle$. Since the Thom class of $MO\langle 8\rangle$ is annihilated by Sq^1, Sq^2, and Sq^4, $H^*BO\langle 8\rangle$ and $H^*MO\langle 8\rangle$ are isomorphic A_2-modules, yielding 2.3 for $H^*MO\langle 8\rangle$.■

We tablulate 2.3 through degree 51, where curved lines indicate Sq^2, straight lines Sq^1, and numbers next to summands the number of copies of the summand. In particular, we note that there are no cyclic A_1-summands beginning in degree 49. This is a key observation in the proof of the following result.

THEOREM 2.8. $H^*MO\langle 8\rangle$ is not a direct sum of cyclic A-modules.

PROOF. $w_8w_{12}w_{14}w_{15}$ is not in $\bar{A}\cdot H^*MO\langle 8\rangle$. [We show that it is not present as a term in any element of $Sq^{2^j} H^*MO\langle 8\rangle$. For $j \geq 4$, this is clear since $H^{17}MO\langle 8\rangle = 0 = H^{33}MO\langle 8\rangle$. For $j = 0$, the only possibility arises from $w_{15} = Sq^1 w_{14}$, but $Sq^1(w_8w_{12}w_{14}w_{14}U) = 0$. A similar argument works for $j = 1$ or 2. The case $j = 3$ is easily exhausted, the only real possibility being $Sq^8(w_{12}w_{14}w_{15}U)$, but $w_8w_{12}w_{14}w_{15}U$ occurs four times here.]

Thus if $H^*MO\langle 8\rangle$ A-splits as a direct sum of cyclics, it must have a summand beginning in 49. The A-splitting induces the A_1-splitting, which we have seen not to contain any cyclic summands beginning in 49.■



Recall that if C is a supHopf algebra of a Hopf algebra B and M is a B-module, then $B//C \otimes M$ and $B \otimes_C M$ are isomorphic B-modules. However, if N is merely a C-module, then $B \otimes_C N$ is a B-module which need not be of the form $B//C \otimes N'$ for any B-module N', nor are $B \otimes_C N$ and $B//C \otimes N$ necessarily isomorphic C-modules. (Consider for example $B = A_2$, $C = A_1$, $N = A_1/(Sq^1)$.) As remarked earlier, it is reasonable to conjecture that $H^*MO\langle 8\rangle \approx A \otimes_{A_2} N$, and the preceding remarks indicate that this carries no implication for its being $A//A_2 \otimes N$, even just as an A_2-module, unless N admits an A-module structure. Nevertheless, it is not unreasonable to try to write $H^*MO\langle 8\rangle$ as $A//A_2 \otimes N$. This can at least be done as stable A_1-modules.

COROLLARY 2.9. As a stable A_1-module $M^*MO\langle 8\rangle$ is isomorphic to

$$A//A_2 \otimes \bigoplus_{S',T'} \Sigma^{4|S'|+4|T'|} J^{|S'|+|T'|} (\Sigma^{-1}I)^{\sum_{t\in T'}(2^{\nu(t)+1}-1)}$$

where

$S' \in$ {nondecreasing sequences of integers $s \geq 4$ such that $\alpha(s-1) > 1$}

$T' \in$ {increasing sequences of integers $2^i + 2^j$ with $0 \leq j < i - 1$}.

PROOF. Similarly to [AP, 3.13] one can show that $A//A_2$ is stably A_1-isomorphic to

$$P(\Sigma^8) \otimes \bigotimes_{\substack{n=2^i 3\\ i\geq 0}} (Z_2 \oplus \Sigma^{4n}(\Sigma^{-1}I)^{2^{\nu(n)+1}-1}J^n).$$

Comparison with (5), and the steps applied to (5) to yield 2.3 yields 2.9.■

An alternate form of 2.9 is

COROLLARY 2.9'. As a stable A_1-module $H^*MO\langle 8\rangle$ is isomorphic to

$$\begin{aligned} &A//A_2 \otimes Z_2[\Sigma^{8i} : i \geq 2] \otimes Z_2[\Sigma^{8i} : i \text{ odd}, \alpha(i-1) > 1] \\ &\otimes \Lambda[\Sigma^{4i}J : i \text{ odd}, \alpha(i-1) > 1] \\ &\otimes \bigotimes_{i\geq 2} \bigoplus_{n\geq 0} \Sigma^{4n(2^i+1)}(\Sigma^{-1}I)^{2n-\alpha(n)}J^n. \end{aligned}$$

If we allow $i = 1$ in the second and last of the five factors, the $A//A_2$ may be deleted, yielding an alternate form of 2.3.

Optimistically, one might hope to extend the A_1-structure on N to an A_2-structure in such a way that $H^*MO\langle 8\rangle \approx A \otimes_{A_2} N$. In order to do this, one will probably need to know the free A_1's.

3. THE SPECTRUM $P_{-\infty}^{\infty} \wedge MO\langle 8\rangle$

Let $P = H^*P_{-\infty}^{\infty}$ denote the A-module $Z_2[x, x^{-1}]$. The calculations of the preceding section lead one to the following conjecture, where notation is as in 2.8.

CONJECTURE 3.1. If $s > 0$ $\mathrm{Ext}_A(P \otimes H^*MO\langle 8\rangle, Z_2)$

$$\approx \bigoplus_{\substack{n\in Z\\ S',T'}} \mathrm{Ext}_{A_1}(\Sigma^{8n-1+4|S'|+4|T'|}J^{|S'|+|T'|}(\Sigma^{-1}I)^{\sum_{t\in T'}(2^{\nu(t)+1}-1)}, Z_2).$$

We could use 2.2 to express this in terms of $\mathrm{Ext}_{A_1}(Z_2, Z_2)$. A stronger conjecture is that this equivalence holds as spectra.

CONJECTURE 3.2. There is an equivalence of "spectra"

$$P_{-\infty}^{\infty} \wedge MO\langle 8\rangle \simeq K \vee \bigvee_{\substack{n\in Z\\ |S'|+|T'| \text{ even}}} \Sigma^{8n-1+4|S'|+4|T'|} bo\langle\Sigma\rangle \vee \bigvee_{\substack{n\in Z\\ |S'|+|T'| \text{ odd}}} \Sigma^{8n-1+4|S'|+4|T'|} bsp\langle\Sigma-1\rangle$$

where K is a wedge of $K(Z_2)$-spectra, and $\Sigma = \Sigma_{t\ T'}(2^{\nu(t)+1}-1)$.

In 3.2, one must be careful what one means by spectrum, since $P_{-\infty}^{\infty}$ is usually defined as an inverse limit of spectra. 3.2 would seem to be quite difficult to prove, probably requiring one to find elements in $KO(P_{-\infty}^{\infty} \wedge MO\langle 8\rangle)$.

The evidence for 3.1 is strong, if not compelling. An attempt to deduce it directly from 2.9 will probably fail, but is plausible for the reasons given below. If, as suggested in the introduction, $H^*MO\langle 8\rangle$ is A-isomorphic to $A \otimes_{A_2} N$ for some A_2-module N, then

$$\mathrm{Ext}_A(P \otimes H^*MO\langle 8\rangle) \approx \mathrm{Ext}_A(A \otimes_{A_2} (P \otimes N)) \approx \mathrm{Ext}_{A_2}(P \otimes N),$$

where the first isomorphism follows from [Liul, 1.7]. One might be tempted by [LDMA, p. 467, last line] to jump from $\mathrm{Ext}_{A_2}(P \otimes N)$ to $\mathrm{Ext}_{A_1}(\bigoplus_{j\in Z} \Sigma^{8j-1}N)$. The author was mildly surprised to discover that this is not a valid deduction. For example, decreasing by 1 the subscript of A_r, $P \otimes A_1/(Sq^1)$ is a free A_1-module, but $A_1/(Sq^1)$ is not a free A_0-module.

The analogous statement for the exotic Singer A-action ([Si : §3]) on $P \otimes N$ is true. It was noted in [AGM] that if N is an A_{r-1}-module then under the Singer action, $P(N)$ is an A_r-module. Mimicking an argument of [AGM] we have

PROPOSITION 3.3. If N is an A_{r-1}module, then

$$\mathrm{Tor}^{A_r}(Z_2, PN) \approx \bigoplus_{k\equiv -1(2^{r+1})} \mathrm{Tor}^{A_{r-1}}(\Sigma^k Z_2, N)$$

and

$$\mathrm{Ext}_{A_r}(PN, Z_2) \approx \bigoplus_{k\equiv -1(2^{r+1})} \mathrm{Ext}_{A_{r-1}}(\Sigma^k N, Z_2).$$

SKETCH OF PROOF. If $0 \longleftarrow N \longleftarrow C_0 \longleftarrow C_1 \longleftarrow C_2 \longleftarrow \ldots$ is an A_{r-1}-resolution, then $0 \longleftarrow PN \longleftarrow PC_0 \longleftarrow PC_1 \longleftarrow PC_2 \longleftarrow$ is an A_r-resolution. [It is exact since $P(\ ) = PA_{r-1} \otimes_{A_{r-1}} (\ )$ -is an exact functor, since PA_{r-1} is a free right A_{r-1}-module.] Thus $\mathrm{Tor}^{A_r}(Z_2, PN)$ is the homology of

$$\begin{array}{ccccc}
\longrightarrow & Z_2 \otimes_{A_r} PC_{i+1} & \xrightarrow{1\otimes Pd} & Z_2 \otimes_{A_r} PC_i & \longrightarrow \\
& \wr\wr & & \wr\wr & \\
\longrightarrow & (Z_2 \otimes_{A_r} PA_{r-1}) \otimes_{A_{r-1}} C_{i+1} & \xrightarrow{1\otimes d} & (Z_2 \otimes_{A_r} PA_{r-1}) \otimes_{A_{r-1}} C_i & \longrightarrow \\
& \wr\wr & & \wr\wr & \\
\longrightarrow & \bigoplus_{k\equiv -1(2^{r+1})} \Sigma^k Z_2 \otimes_{A_{r-1}} C_{i+1} & \xrightarrow{1\otimes d} & \bigoplus_{k\equiv -1(2^{r+1})} \Sigma^k Z_2 \otimes_{A_{r-1}} C_i & \longrightarrow .
\end{array}$$ ■

The difficulty is that, unlike the situation for A-modules, $P(N)$ and $P \otimes N$ need not be isomorphic A_r-modules if N is an A_r-module.

One might try to prove 3.1 directly by filtering $P \otimes H^*MO\langle 8\rangle$ with F_j the A-module spanned by $x^k \otimes m$ for $k < 8j - 1$, and finding classes in $P \otimes H^*MO\langle 8\rangle/F_j$ which give the splitting of 3.1 for $n \geq j$. This has been done through 22 degrees but the pattern is not clear.

We expand upon the discussion in [DM1] of the possible usefulness of $P_{-\infty}^{\infty} \wedge MO\langle 8\rangle$ in obstruction theory by proving

PROPOSITION 3.4. There is a map

$$BO\langle 8\rangle \longrightarrow \Sigma P_{-\infty}^{\infty} \wedge MO\langle 8\rangle$$

which then followed by the natural map into $\Sigma P_N \wedge MO\langle 8\rangle$ gives the MO<8>-orientation of [DGIM].

PROOF. Dualize the composite

$$BO_{8N}\langle 8\rangle \wedge RP^{k-1} \longrightarrow BO_{8N}\langle 8\rangle \xrightarrow{\gamma\otimes\xi} MO_{8N}\langle 8\rangle \longrightarrow \Sigma^{8N} MO\langle 8\rangle$$

to obtain $BO_{8N}\langle 8\rangle \longrightarrow \Sigma^{8N+1} P_{-k}^{-2} \wedge MO\langle 8\rangle$. Passing to $\varprojlim_{k}$ and using the equivalence of [DGIM, 1.2] we obtain

$$BO_{8N}\langle 8\rangle \longrightarrow P_{-\infty}^{8N-2} \wedge MO\langle 8\rangle.$$

The desired map is obtained by passing to $\varinjlim_{N}$. ∎

Conjecture 3.2 and Proposition 3.4 might be used to restrict the possible maps $X \longrightarrow BO\langle 8\rangle \longrightarrow \Sigma P_N \wedge MO\langle 8\rangle$.

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September 30, 1981