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Preface

Differential forms are a powerful computational and theoretical tool. They play a central role in mathematics, in such areas as analysis on manifolds and differential geometry, and in physics as well, in such areas as electromagnetism and general relativity. In this book, we present a concrete and careful introduction to differential forms, at the upper-undergraduate or beginning graduate level, designed with the needs of both mathematicians and physicists (and other users of the theory) in mind.

On the one hand, our treatment is concrete. By that we mean that we present quite a bit of material on how to do computations with differential forms, so that the reader may effectively use them.

On the other hand, our treatment is careful. By that we mean that we present precise definitions and rigorous proofs of (almost) all of the results in this book.

We begin at the beginning, defining differential forms and showing how to manipulate them. First we show how to do algebra with them, and then we show how to find the exterior derivative $d\varphi$ of a differential form φ. We explain what differential forms really are: Roughly speaking, a k-form is a particular kind of function on k-tuples of tangent vectors. (Of course, in order to make sense of this we must first make sense of tangent vectors.) We carry on to our main goal, the Generalized Stokes's Theorem, one of the central theorems of mathematics. This theorem states:

THEOREM (*Generalized Stokes's Theorem (GST)*). *Let M be an oriented smooth k-manifold with boundary ∂M (possibly empty) and let ∂M be given the induced orientation. Let φ be a $(k-1)$-form on M with compact support. Then*

$$\int_M d\varphi = \int_{\partial M} \varphi.$$

This goal determines our path. We must develop the notion of an oriented smooth manifold and show how to integrate differential forms on these. Once we have done so, we can state and prove this theorem.

The theory of differential forms was first developed in the early twentieth century by Elie Cartan, and this theory naturally led to de Rham cohomology, which we consider in our last chapter.

One thing we call the reader's attention to here is the theme of "naturality" that pervades the book. That is, everything commutes with pull-backs–this cryptic statement will become clear upon reading the book—and this enables us to do all our calculations on subsets of $\mathbb{R}^n$, which is the only place we really know how to do calculus.

This book is an outgrowth of the author's earlier book *Differential Forms: A Complement to Vector Calculus*. In that book we introduced differential forms at a lower level, that of third semester calculus. The point there was to show how the theory of differential forms unified and clarified the material in multivariable calculus: the gradient of a function, and the curl and divergence of a vector field (in $\mathbb{R}^3$) are all "really" special cases of the exterior derivative of a differential form, and the classical theorems of Green, Stokes, and Gauss are all "really" special cases of the GST. By "really" we mean that we must first recast these results in terms of differential forms, and this is done by what we call the "Fundamental Correspondence."

However, in the (many) years since that book appeared, we have received a steady stream of emails from students and teachers who used this book, but almost invariably at a higher level. We have thus decided to rewrite it at a higher level, in order to address the needs of the actual readers of the book. Our previous book had minimal prerequisites, but for this book the reader will have to be familiar with the basics of point-set topology, and to have had a good undergraduate course in linear algebra. We use additional linear algebra material, often not covered in such a course, and we develop it when we need it.

We would like to take this opportunity to correct two historical errors we made in our earlier book. One of the motivations for developing vector calculus was, as we wrote, Maxwell's equations in electromagnetism. We wrote that Maxwell would have recognized vector calculus. In fact, the (common) expression of those equations in vector calculus terms was not due to him, but rather to Heaviside.

But it is indeed the case that this is a nineteenth century formulation, and there is an illuminating reformulation of Maxwell's equations in terms of differential forms (which we urge the interested reader to investigate). Also, Poincaré's work in celestial mechanics was another important precursor of the theory of differential forms, and in particular he proved a result now known as Poincaré's Lemma. However, there is considerable disagreement among modern authors as to what this lemma is (some say it is a given statement, others its converse). In our earlier book we wrote that the statement in one direction was Poincaré's Lemma, but we believe we got it backwards then (and correct now). See Remark 1.4.2.

We conclude with some remarks about notation and language. Results in this book have three-level numbering, so that, for example, Theorem 1.2.7 is the 7$^{\text{th}}$ numbered item in Chapter 1, Section 2. The ends of proofs are marked by the symbol □. The statements of theorems, corollaries, etc., are in italics, so are clearly delineated. But the statements of definitions, remarks, etc., are in ordinary type, so there is nothing to delineate them. We thus mark their ends by the symbol ◇. We use $A \subseteq B$ to mean that A is a subset of B, and $A \subset B$ to mean that A is a proper subset of B. We use the term "manifold" to mean precisely that, i.e., a manifold without boundary. The term "manifold with boundary" is a generalization of the term "manifold," i.e., it includes the case when the boundary is empty, in which case it is simply a manifold.

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