Preface

The title of this book, *Fundamentals of Algebraic Topology*, summarizes its aims very well.

In writing this book we have attempted to provide the reader with a guide to the fundamental results of algebraic topology, but we have not attempted to provide an exhaustive treatment.

Our choice of topics is quite standard for an introductory book on algebraic topology, but a description of our approach is in order.

We begin with a short introductory chapter, with basic definitions. We assume the reader is already familiar with basic notions from point-set topology, and take those for granted throughout the book.

We then devote Chap. 2 to the fundamental group, including a careful discussion of covering spaces, van Kampen's theorem, and an application of algebraic topology to obtain purely algebraic results on free groups. In general, algebraic topology involves the use of algebraic methods to obtain topological information, but this is one instance in which the direction is reversed.

We then move on to discuss homology and cohomology. Here we follow the axiomatic approach pioneered by Eilenberg and Steenrod. In Chap. 3 we introduce the famous Eilenberg-Steenrod axioms. Indeed, in this chapter we consider arbitrary generalized homology theories and derive results that hold for all of them. Then, in Chap. 4, we specialize to ordinary homology theory with integer coefficients, and derive results and applications in this situation, still proceeding axiomatically. Of particular note are such important results as the Brouwer fixed point theorem and invariance of domain, which follow from the existence of a homology theory, not from the details of its particular construction. Also of particular note is our introduction of CW complexes and our development of cellular homology, again from the axioms.

Of course, at some point we must show that a homology theory actually exists, and we do that in Chap. 5, where we construct singular homology. We deal with the full panoply here – homology, cohomology, arbitrary coefficients, the Künneth formula, products, and duality.

Manifolds are a particularly important class of topological spaces, and we devote Chap. 6 to their study.

Finally, in Chap. 7 we give a short introduction to homotopy theory.

Arguments in algebraic topology involve a mixture of algebra and topology. But some arguments are purely algebraic, and, indeed, algebraic topology spawned a new branch of algebra, homological algebra, to deal with the algebraic issues it raised. While it is not always possible to completely separate the topology and the algebra, in many instances it is. In those instances, we find it advantageous to do so, as it better reveals the logical structure of the subject. Thus we have included the basic algebraic constructions and results in an appendix, rather than mixing them in with the rest of the text.

In our discussion of Poincaré duality on manifolds, we need some basic facts about bilinear forms, and we summarize them in a second appendix.

Algebraic topology also spawned the language of categories and functors. We have tended to avoid this language in the text, as it is mostly (though not entirely) superfluous for our purposes here. But it is illuminating language, and essential for students who wish to go further, and so we have included a third appendix that introduces it.

There are several points we wish to call the reader's attention to. The first is that we have not felt compelled to give the proofs of all the theorems. To be sure, we give most of them (and leave a few of them as exercises for the reader), but we have omitted some that are particularly long or technical. For example, we have not proved van Kampen's theorem, nor have we proved that singular homology satisfies the excision axiom. The second is that we have not always stated results in the maximum generality. For example, in developing products in homology and cohomology we have restricted the pairs of spaces involved, and the coefficients of the (co)homology groups, to the situations in which they are most often used in practice.

The third is that we have hewed to the axiomatic foundations of the subject. Since its inception over a century ago, algebraic topology has built a vast superstructure on these foundations, a superstructure we hope the student will go on to investigate. But we do not investigate it here. For example, we say very little about techniques for computing homotopy groups.

However, we provide a short bibliography where the reader can find the material we omit, as well as material that is beyond our scope. (In other words, the reader who wants to see this, or wants to study further in algebraic topology, should consult these books.)

Our notation and numbering scheme here are rather standard, and there is little to be said. But we do want to point out that we use $A \subseteq B$ to mean that A is a subset of B and $A \subset B$ to mean that A is a proper subset of B.

Lemmas, propositions, theorems, and corollaries are stated in italics, which clearly delimits them from the text that follows. Similarly, proofs are delimited by the symbol $\square$ at the end. But there is usually nothing to delimit definitions, examples, and remarks, which are stated in roman. We use the symbol $\Diamond$ at the end of these for that purpose.

Bethlehem, PA, USA Steven H. Weintraub
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