VECTOR FIELDS ON $RP^m \times RP^n$

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ABSTRACT. The span of a manifold is its maximum number of linearly independent vector fields. We discuss the question, still unresolved, of whether $\mathrm{span}(P^m \times P^n)$ always equals $\mathrm{span}(P^m) + \mathrm{span}(P^n)$. Here P^n denotes real projective space. We use BP-cohomology to obtain new upper bounds for $\mathrm{span}(P^m \times P^n)$, much stronger than previously known bounds.

1. INTRODUCTION

The **span** of a manifold is the maximum number of linearly independent vector fields on it. The following result is well-known. Throughout the paper, $\nu(n)$ denotes the exponent of 2 in n, and P^n denotes real projective space.

Theorem 1.1. *Let $V(n) = 8a + 2^b - 1$ if $\nu(n+1) = 4a + b$ with $0 \le b \le 3$. Then* $\mathrm{span}(P^n) = V(n)$.

Proof. It was proved by Adams in [1] that $\mathrm{span}(S^n) = V(n)$. Since linearly independent (l.i.) vector fields on P^n pull back to l.i. vector fields on S^n, this implies $\mathrm{span}(P^n) \le V(n)$. Since the vector fields on S^n can be chosen to satisfy $v_i(-x) = -v_i(x)$, they induce l.i. vector fields on P^n. ([2, p.140]) $\qquad\square$

This of course implies that

$$(1.2) \qquad \mathrm{span}(P^m \times P^n) \ge V(m) + V(n).$$

Although it seems unlikely that equality always holds in (1.2), there are no known examples in which $\mathrm{span}(P^m \times P^n)$ exceeds $V(m) + V(n)$. Our first result shows that equality does hold in (1.2) in many cases.

Theorem 1.3. *If neither* $m + 1$ *nor* $n + 1$ *is divisible by 16, or if* $m = 1$, 3, *or* 7, *then*

$$\mathrm{span}(P^m \times P^n) = V(m) + V(n).$$

The proof of this, which is quite elementary, is given in Section 2.

Our second result uses BP-cohomology to obtain new upper bounds for $\mathrm{span}(P^m \times P^n)$ which are exponentially stronger than previously known results. This should be considered the main result of the paper.

Theorem 1.4. *Let* $r = \nu(M) \geq 4$ *and* $t = \nu(N) \leq r$. *Then*

$$\mathrm{span}(P^{2M-1} \times P^{2N-1}) \leq \begin{cases} 14 \cdot 2^e - 4 \\ 18 \cdot 2^e - 4 \\ 20 \cdot 2^e + 8k + 6 \\ 24 \cdot 2^e + 4k - 2 \end{cases}$$

$$if \quad r + t = 2^e + 2e + \begin{cases} 2 \\ 3 \\ 4 + k, & 0 \leq k < 2^e - 2 \\ 4 + k, & k = 2^e - 2, 2^e - 1 \end{cases} \quad and \quad t > \begin{cases} e \\ e + 1 \\ e \\ e. \end{cases}$$

Here $e \geq 1$ *or* $(r, t, e, k) = (4, 1, 0, 0)$.

We prove this result in Section 2. In Section 3, we give numerical illustrations of this theorem and compare it with previous results. We also discuss some cases in which it can be extended.

2. Proofs

In this section, we prove Theorems 1.3 and 1.4. The following result is well-known.

Proposition 2.1. $\mathrm{span}(P^m \times P^n) \leq 2^{\nu(m+1)} + 2^{\nu(n+1)} - 2$.

Proof. The total Stiefel-Whitney class of $P^m \times P^n$ is $(1+x_1)^{m+1}(1+x_2)^{n+1}$, truncated after x_1^m and x_2^n. By well-known properties of binomial coefficients mod 2, the highest nonzero Stiefel-Whitney class is $x_1^{m+1-2^{\nu(m+1)}} x_2^{n+1-2^{\nu(n+1)}}$. Thus the tangent bundle cannot be stably equivalent to a bundle of dimension less than

$$d := m + 1 - 2^{\nu(m+1)} + n + 1 - 2^{\nu(n+1)}.$$

Hence the number of l.i. vector fields is at most $m + n - d$, as claimed.   □

Now we can prove our first theorem.

Proof of Theorem 1.3. The first case follows from the previous proposition together with the fact that $V(n) = 2^{\nu(n+1)} - 1$ if $n \not\equiv 15 \mod 16$. For the second case, we need the important notion of *stable span*. The span of a vector bundle θ is its maximal number of l.i. sections, and the stable span of a manifold M, denoted $\mathrm{span}^0(M)$, equals $\mathrm{span}(\tau(M) + m\epsilon) - m$ for $m > 0$, which is easily seen to be independent of such m. Here τ denotes the tangent bundle, and ϵ a trivial bundle.

The restriction of $\tau(P^m \times P^n)$ to $* \times P^n$ is $\tau(P^n) + m\epsilon$. Hence

$$\mathrm{span}(P^m \times P^n) \leq m + \mathrm{span}^0(P^n).$$

For $m = 1$, 3, or 7, $m = \mathrm{span}(P^m)$ and so it remains to show that $\mathrm{span}^0(P^n) = \mathrm{span}(P^n)$.

If n is even, both are 0 since $w_n(\tau(P^n)) \neq 0$. In [3, 1.11], James and Thomas proved that if n is odd, an n-plane bundle over P^n is equivalent to $\tau(P^n)$ if and only if they are stably equivalent, implying the result in this case. $\square$

In the rest of this section, we prove our second theorem, which is more substantial. Since $\tau(P^{2M-1} \times P^{2N-1}) + 2\epsilon \approx 2M\xi_{2M-1} \times 2N\xi_{2N-1}$, then, using the BP-Euler class as in [6, pp.331-332], we obtain that if $\mathrm{span}(P^{2M-1} \times P^{2N-1}) > s$, then

(2.2)
$$\sum_{i,j}(-1)^{i+j}\binom{M}{i}\binom{N}{j}x_1^i x_2^j x_3^{M+N-i-j} = 0 \in BP^{2M+2N}(P^{2M-2} \times P^{2N-2} \times P^{s+2}).$$

Note that we have restricted to $P^{2M-2} \times P^{2N-2} \subset P^{2M-1} \times P^{2N-1}$ to simplify the calculation. Here x_i is a 2-dimensional class corresponding to the ith factor, and BP is the 2-local Brown-Peterson spectrum. The conclusion (2.2) also holds with Cartesian product replaced by smash product, using the direct sum splitting.

It will be convenient to work with the Johnson-Wilson spectrum $BP\langle 3 \rangle$. See, for example, [4, p.117]. We will call if B. Recall that $B_* = \mathbb{Z}_{(2)}[v_1, v_2, v_3]$ with $|v_i| = 2^{i+1} - 2$. Let $v_0 = 2$ and let I denote the ideal (v_0, v_1, v_2, v_3). There is a power series $[2](x) = \sum a_j x^{j+1}$ with $a_j \in B_{2j}$ satisfying $a_0 = 2$ and

$$[2](x) \equiv v_0 x + v_1 x^2 + v_2 x^4 + v_3 x^8 \mod I^2.$$

See, for example, [4, p.120]. Let $P_1 = P^\infty$ and $B^* = B_{-*}$. Gysin sequence arguments show that

$$B^*(P^{2n-2}) \approx B^*[x]/(x^n, [2](x))$$

and

$$B_*(P_1) \approx B_*(z_i : i \geq 1)/(\sum a_j z_{i-j}),$$

where $|z_i| = 2i - 1$, and there are duality isomorphisms

(2.3) $$B^{2n-2i}(P^{2n-2}) \approx B_{2i-1}(P_1)$$

for $i < n$ under which x^{n-i} corresponds to z_i. These isomorphisms pass to 2- and 3-fold smash products, as described briefly in [6, p.330]. If i, j, and k are positive integers, let $[i, j, k] \in B_{2(i+j+k)-3}(P_1 \wedge P_1 \wedge P_1)$ denote the external product of classes z_i, z_j, and z_k, as in [4, p.120]. Let

$$Q = B_*(P_1) \otimes_{B_*} B_*(P_1) \otimes_{B_*} B_*(P_1).$$

By [4, 1.3], Q is a subgroup of $B_*(P_1 \wedge P_1 \wedge P_1)$. This Q is a B_*-module. Let $F_s = I^s \cdot Q \subset Q$. This filtration could be thought of as filtration in an Adams spectral sequence. The content of [4, 2.3] can be restated as follows.

Proposition 2.4. *For all $s \geq 0$, F_s/F_{s+1} is a graded $\mathbb{Z}_2$-vector space with basis all $v_3^s[i_1, i_2, i_3]$, $i_j > 0$.*

The action of $v_0 : F_0/F_1 \to F_1/F_2$ can be determined using the 2-series $[2](x)$ in the following way. Temporarily write $[e_1, e_2, e_3]$ as $z_1^{e_1} z_2^{e_2} z_3^{e_3}$. The 2-series forces relations

$$(v_0 + v_1 z_i^{-1} + v_2 z_i^{-3} + v_3 z_i^{-7}) z_1^{e_1} z_2^{e_2} z_3^{e_3} = 0 \text{ in } F_1/F_2.$$

We apply the relation to repeatedly replace v_0 by $v_1 z_1^{-1} + v_2 z_1^{-3} + v_3 z_1^{-7}$, v_1 by $v_0 z_2 + v_2 z_2^{-2} + v_3 z_2^{-6}$, and v_2 by $v_0 z_3^3 + v_1 z_3^2 + v_3 z_3^{-4}$. This is the approach that was taken in [6]. For $i = 0$, 1, and 2, we obtain infinite series $f_i(z_1, z_2, z_3)$ such that $v_i = v_3 \cdot f_i(z_1, z_2, z_3)$.

For example, the determination of f_0 begins as follows.

$$
\begin{aligned}
v_0 &= v_1 z_1^{-1} + v_2 z_1^{-3} + v_3 z_1^{-7} \\
&= (v_0 z_2 + v_2 z_2^{-2} + v_3 z_2^{-6}) z_1^{-1} \\
&\quad + (v_0 z_3^3 + v_1 z_3^2 + v_3 z_3^{-4}) z_1^{-3} + v_3 z_1^{-7} \\
&= (v_1 z_1^{-1} + v_2 z_1^{-3} + v_3 z_1^{-7})(z_1^{-1} z_2 + z_1^{-3} z_3^3) \\
&\quad + (v_0 z_2 + v_2 z_2^{-2} + v_3 z_2^{-6}) z_1^{-3} z_3^2 \\
&\quad + (v_0 z_3^3 + v_1 z_3^2 + v_3 z_3^{-4}) z_1^{-1} z_2^{-2} \\
&\quad + v_3 (z_1^{-7} + z_1^{-1} z_2^{-6} + z_1^{-3} z_3^{-4}) \\
&= \cdots
\end{aligned}
$$

The three monomials in the last line above could be thought of as the start of f_0. However, they may (and will) be cancelled later in the algorithm. The procedure does converge in the sense that any monomial can only appear a finite number of times. This is true because every step decreases $3\nu_{z_1} + \nu_{z_2} - 2\nu_{v_0} + \nu_{v_2}$ in all monomials.

The series f_0, f_1, and f_2 must satisfy the equations

$$
\begin{aligned}
f_0 &= f_1 z_1^{-1} + f_2 z_1^{-3} + z_1^{-7} \\
f_1 &= f_0 z_2 + f_2 z_2^{-2} + z_2^{-6} \\
f_2 &= f_0 z_3^3 + f_1 z_3^2 + z_3^{-4}.
\end{aligned}
$$

Rearrange them so that they appear as a system over $\mathbb{Z}_2$ for the three unknowns f_0, f_1, and f_2, and solve by Cramer's rule to obtain

$$
\begin{aligned}
f_0 &= \frac{z_1^{-1} z_2^{-2} z_3^{-4} + z_1^{-1} z_2^{-6} + z_1^{-3} z_2^{-6} z_3^2 + z_1^{-3} z_3^{-4} + z_1^{-7} + z_1^{-7} z_2^{-2} z_3^2}{1 + z_2^{-2} z_3^2 + z_1^{-1} z_2 + z_1^{-1} z_2^{-2} z_3^3 + z_1^{-3} z_2 z_3^2 + z_1^{-3} z_3^3} \\
&= z_1^{-1} z_2^{-2} z_3^{-4} + z_1^{-1} z_2^{-4} z_3^{-2} + z_1^{-2} z_2^{-1} z_3^{-4} + z_1^{-2} z_2^{-4} z_3^{-1} + z_1^{-4} z_2^{-1} z_3^{-2} + z_1^{-4} z_2^{-2} z_3^{-1}.
\end{aligned}
$$

The latter equation can be verified by cross multiplication. Note that it turns out that f_0 is not an infinite series after all. Our conclusion is stated in the next two results.

Theorem 2.5. *For distinct nonnegative integers i, j, and k, let $\mathcal{P}(i, j, k)$ denote the set consisting of the six permutations of i, j, and k. Then, in Q,*

$$
2[e_1, e_2, e_3] \equiv v_3 \sum_{(i,j,k) \in \mathcal{P}(1,2,4)} [e_1 - i, e_2 - j, e_3 - k] \mod F_2.
$$

Iterating this, we obtain

Corollary 2.6. *First,*

$$2^{2^t}[e_1, e_2, e_3] \equiv v_3^{2^t} \sum [e_1 - i, e_2 - j, e_3 - k] \quad \mod F_{2^t+1},$$

where the sum is taken over all $(i, j, k) \in \mathcal{P}(2^t, 2^{t+1}, 2^{t+2})$. *More generally, if* $m = \sum_{\ell=1}^{d} 2^{t_\ell}$ *with* $\{t_\ell\}$ *distinct, then*

$$2^m[e_1, e_2, e_3] \equiv v_3^m \sum [e_1 - i_1 - \cdots - i_d, e_2 - j_1 - \cdots - \jmath_d, e_3 - k_1 - \cdots - k_d] \quad \mod F_{m+1},$$

summed over all $(i_\ell, j_\ell, k_\ell) \in \mathcal{P}(2^{t_\ell}, 2^{t_\ell+1}, 2^{t_\ell+2})$ *with* $\ell = 1, \ldots, d$.

The implication of $\mathrm{span}(P^{2M-1} \times P^{2N-1}) > 2s' - 4$ in (2.2) becomes

$$\sum (-1)^{i+j} \binom{M}{M-i} \binom{N}{N-j} x_1^{M-i} x_2^{N-j} x_3^{s'-(s'-i-j)} = 0 \in B^*(P^{2M-2} \wedge P^{2N-2} \wedge P^{2s'-2})$$

and then, under the isomorphism (2.3),

$$(2.7) \qquad \sum (-1)^{i+j} \binom{M}{i} \binom{N}{j} [i, j, s' - i - j] = 0 \in Q.$$

We preview the detailed proof of Theorem 1.4 by illustrating with the case $r = 5$, $t = 3$. This is the fourth case in the theorem, with $e = 1$ and $k = 0$. The claim then is that if $M \equiv 32 \mod 64$ and $N \equiv 8 \mod 16$, then $\mathrm{span}(P^{2M-1} \times P^{2N-1}) \leq 46$. To prove this, we assume $\mathrm{span}(P^{2M-1} \times P^{2N-1}) > 46$ and deduce as above that

$$(2.8) \qquad \sum (-1)^{i+j} \binom{64k+32}{i} \binom{16\ell+8}{j} [i, j, 25 - i - j] = 0 \in Q.$$

Since all nonzero terms have $i + j \leq 24$, all terms in the sum are divisible by 2^2, and so our sum is in F_2. Here and later we use that $\nu\binom{u2^r}{i} \geq r - \nu(i)$, with equality if $i \leq 2^r$. Note that the filtration-1 term $2[16, 8, 1]$ which occurs in (2.8) is 0 since subtracting a permutation of $(4, 2, 1)$ from $(16, 8, 1)$ always results in a non-positive entry.

The only terms in (2.8) not divisible by 2^3 are $2^2 u_1[16, 4, 5]$ and $2^2 u_2[8, 8, 9]$ with u_i odd, and so, mod F_3, our sum equals $2^2[16, 4, 5] + 2^2[8, 8, 9]$. By Corollary 2.6, this is equal, mod F_3, to

$$v_3^2([16-8, 4-2, 5-4] + [8-2, 8-4, 9-8] + [8-4, 8-2, 9-8]) = v_3^2([8, 2, 1] + [6, 4, 1] + [4, 6, 1]),$$

which is nonzero by 2.4, contradicting the assumption that the span is greater than 46.

Now we begin the proof of Theorem 1.4 in earnest. For the first case in 1.4, we assume $\mathrm{span}(P^{2M-1} \times P^{2N-1}) > 14 \cdot 2^e - 4$. Then, with u_i odd,

$$(2.9) \qquad \sum_{i,j} \pm \binom{u_1 2^r}{i}\binom{u_2 2^t}{j}[i, j, 7 \cdot 2^e - i - j] = 0.$$

Here the pair (r, t) appears in the list

$$(2^e + e + 1, e + 1), \ldots, (2^{e-1} + e + 1, 2^{e-1} + e + 1),$$

always summing to $2^e + 2e + 2$. Since we must have $i + j < 7 \cdot 2^e$, the terms in the sum having lowest filtration are $T_1 := 2^{2^e-1}[2^{e+2}, 2^{e+1}, 2^e]$ and, if $t > e + 1$, $T_2 := 2^{2^e-1}[2^{e+1}, 2^{e+2}, 2^e]$. By Corollary 2.6, the term T_1 equals, mod F_{2^e},

$$v_3^{2^e-1}[2^{e+2} - (2^{e+2} - 4), 2^{e+1} - (2^{e+1} - 2), 2^e - (2^e - 1)] = v_3^{2^e-1}[4, 2, 1]$$

plus perhaps $v_3^{2^e-1}$ times other terms of degree $2 \cdot 7$. The term T_2 equals, mod F_{2^e}, the sum of terms

$$v_3^{2^e-1}[2^{e+1} - \sum_{i=0}^{e-1} 2^{i+a_i}, 2^{e+2} - \sum_{i=0}^{e-1} 2^{i+b_i}, 2^e - \sum_{i=0}^{e-1} 2^{i+c_i}],$$

where each (a_i, b_i, c_i) is a permutation of $(0, 1, 2)$. We must have $\sum 2^{i+b_i} \leq 2^{e+2} - 4$, and hence T_2 has no terms of the form $v_3^{2^e-1}[-, 2, -]$. Thus the nonzero term $v_3^{2^e-1}[4, 2, 1]$ in T_1 is uncancelled in filtration $2^e - 1$, and so the LHS of (2.9) is nonzero, a contradiction.

For the second case, if span $> 18 \cdot 2^e - 4$, then

$$(2.10) \qquad \sum \pm \binom{u_1 2^r}{i}\binom{u_2 2^t}{j}[i, j, 9 \cdot 2^e - i - j] = 0.$$

Under the hypotheses, the only term of lowest filtration is $2^{2^e-1}[2^{e+2}, 2^{e+2}, 2^e]$, and this equals, mod F_{2^e}, $v_3^{2^e-1}[2^{e+1} + 2, 4, 1]$ plus other terms, and hence is nonzero.

The third and fourth cases are distinguished by which has nonzero terms in the smaller grading. For the fourth case, the terms of smallest filtration are

$$2^{2^e+k}[2^{e+3}, 2^{e+1}, 2^{e+1} + 1 + 2k],$$

possibly $2^{2^e+k}[2^{e+1}, 2^{e+3}, 2^{e+1} + 1 + 2k]$, and if $k = 0$ then also $2^{2^e}[2^{e+2}, 2^{e+2}, 2^{e+2} + 1]$. The first term equals, mod F_{2^e+k+1}, $v_3^{2^e+k}[2^{e+2} - 4k, 2^e - k, 1]$ plus possibly other terms, and this cannot be cancelled by either of the others.

Under the hypothesis $k < 2^e - 2$ of the third case, there is a term

$$2^{2^e + k + 1}[2^{e+2}, 2^{e+1}, 2^{e+2} + 4k + 5]$$

of smaller grading than the terms just considered, and in this grading there are no terms of smaller filtration. If $t > e + 1$, there is also a nonzero term

$$2^{2^e + k + 1}[2^{e+1}, 2^{e+2}, 2^{e+2} + 4k + 5].$$

The first term equals, mod higher filtration and other terms,

$$T := v_3^{2^e + k + 1}[2^{e+1} - 2k - 2, 2^e - k - 1, 1].$$

When the second term is rewritten as a sum of terms

$$v_3^{2^e + k + 1}[2^{e+1} - \sum i_\ell, 2^{e+2} - \sum j_\ell, 2^{e+2} + 4k + 5 - \sum k_\ell]$$

as in 2.6, the only way it could contain the term T is if $\sum k_\ell = 2^{e+2} + 4k + 4$. Then $\sum j_\ell \le 2^{e+1} + 2k + 2$, and since $k < 2^e - 2$, we must have $2^{e+2} - \sum j_\ell > 2^e - k - 1$, and so T cannot be cancelled.

3. NUMERICAL RESULTS

Theorem 1.4 is exponentially better than the previous best known results, most of which are given in Proposition 2.1. For a typical example, if $\nu(m + 1) = \nu(n + 1) = 2^{e-1} + e + 2$, then we obtain

$$\mathrm{span}(P^m \times P^n) \le 2^{e+4} - 2^{e+1} - 4,$$

while the result from 2.1 is

$$\mathrm{span}(P^m \times P^n) \le 2^{2^{e-1} + e + 3} - 2.$$

In [7], K-theoretic methods were used to obtain nonexistence results for vector fields on products of real projective spaces. In [5], slight improvements were obtained in some cases, but not in any applicable to this paper. All these results are weaker than those of Proposition 2.1 unless both $m + 1$ and $n + 1$ are 2-powers, or $m + 1$ is a 2-power and $n < m/2$. The K-theory bound is always more than $1/2$ times the Stiefel-Whitney bound. See Table 3.2 for numerical examples. In our example in the previous paragraph, the K-theoretic methods give no new information, compared to Proposition 2.1, except in the case of $P^m \times P^m$ with $m = 2^{2^{e-1} + e + 2} - 1$, and in this

case the bound is roughly $3 \cdot 2^{2e-1+e+1} + 2^{e-1} + e - 1$, still exponentially larger than our bound.

Theorem 1.4 can be extended to include cases in which t-values are smaller than they are in that theorem. We did not include them there because their patterns become too complicated. We won't even list many of them here because, for best results, they begin to involve more than just the 2-divisibility of N. The following result describes what we can deduce when t is 1 (resp. 1 or 2) smaller than is allowed in the first (resp. second) case of Theorem 1.4.

Proposition 3.1. *Let $e \geq 1$, $r = \nu(M)$, and $t = \nu(N)$. Then*

$$\operatorname{span}(P^{2M-1} \times P^{2N-1})$$
$$\leq \begin{cases} 20 \cdot 2^e - 2 & \text{if } r = 2^e + e + 2 \text{ and } t = e + 1 \text{ or } (e > 1 \text{ and } N \equiv 3 \cdot 2^e \bmod 2^{e+2}) \\ 22 \cdot 2^e - 6 & \text{if } r = 2^e + e + 2 \text{ and } t = e = 1 \text{ or } N \equiv 2^e \bmod 2^{e+2} \\ 20 \cdot 2^e + 6 & \text{if } r = 2^e + e + 3 \text{ and } N \equiv 3 \cdot 2^e \bmod 2^{e+2} \\ 38 \cdot 2^e - 6 & \text{if } r = 2^e + e + 3 \text{ and } t = e. \end{cases}$$

Note that the third case here is a strengthening of the fourth case which applies when more information about N is known other than just its 2-exponent. Other similar strengthenings can be given, but become too tedious to list.

Proof. These are proved by the same method as the proof of 1.4. The determining terms of lowest filtration in the four cases here are

$$2^{2^e}[2^{e+2}, 2^{e+1}, 2^{e+2} + 1] \quad \sim \quad v_3^{2^e}[2^{e+1}, 2^e, 1]$$
$$2^{2^e - 1}[2^{e+3}, 2^e, 2^{e+1} - 1] \quad \sim \quad v_3^{2^e - 1}[2^{e+2} + 4, 1, 1]$$
$$2^{2^e + 1}[2^{e+2}, 2^{e+1}, 2^{e+2} + 5] \quad \sim \quad v_3^{2^e + 1}[2^{e+1} - 2, 2^e - 1, 1]$$
$$2^{2^e - 1}[2^{e+4}, 2^e, 2^{e+1} - 1] \quad \sim \quad v_3^{2^e - 1}[3 \cdot 2^{e+2} + 4, 1, 1].$$

$\square$

We close with a table comparing the various bounds for $\operatorname{span}(P^m \times P^{111})$ for $m = 2^e - 1$ and $m = 3 \cdot 2^e - 1$ for $5 \leq e \leq 17$. The first column (after the e-column) gives the lower bound for $\operatorname{span}(P^m \times P^{111})$ given by (1.2). It is the same for $m = 2^e - 1$ and $m = 3 \cdot 2^e - 1$. The second column gives our upper bound for $\operatorname{span}(P^m \times P^{111})$ from 1.4, 3.1, and a slight extension of the third case of 3.1. In this range, it is the same for either value of m. The third column gives the upper bound given by

Stiefel-Whitney classes (Proposition 2.1). It is always the same for the two m-values. The final column gives the upper bound obtained by Suzuki in [7] when $m = 2^e - 1$. If $m = 3 \cdot 2^e - 1$, his bound is larger than the Stiefel-Whitney bound, and so we do not bother to list it.

Table 3.2. *Bounds for* $\text{span}(P^m \times P^{111})$*,* $m = 2^e - 1$ *or* $3 \cdot 2^e - 1$

e	*lower bound*	*our upper bound*	*Stief-Whit upper bound*	*Suzuki upper bound* $m = 2^e - 1$
5	17	32	46	130
6	19	46	78	148
7	23	50	142	182
8	24	52	270	246
9	25	78	526	374
10	27	86	1038	630
11	31	94	2062	1146
12	32	102	4110	2170
13	33	106	8206	4218
14	35	158	16398	8316
15	39	166	32782	16510
16	40	174	65550	32894
17	41	182	131086	65662

Note that our upper bound is moderately close to the known lower bound. For fairly large e, on the other hand, the other known upper bounds are exponentially larger than ours and the lower bound.

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