1. $1/(\frac{1}{2} - \frac{1}{3}) =$

2. If $\sqrt{n^3 + n^3 + n^3 + n^3 + n^3} = 25$, then $n =$

3. If prices go down by 20%, by what percentage does your purchasing power increase? ("Purchasing power" means the amount of goods that you can purchase for a fixed amount of money.)

4. What is the ratio of the area of a circle to the area of an inscribed square?

5. In a certain football league, the only ways to score points are to kick a field goal for 3 points or score a touchdown for 7 points. How many positive scores cannot possibly be obtained by a team in this game?

6. In triangle ABC, D lies on AC, and E lies on BC, with DE parallel to AB. If $AC = 5$, $DC = 4$, and $BE = 2$, then what is the length of EC?

7. How many right triangles have all side lengths integers with both leg lengths prime numbers?

8. The cross-sectional area of a tree is a linear function of time. The radius is 2 feet in 1910 and 3 feet in 1985. What is its radius in 2000?

9. Let $f(x) = (x^5 - 1)(x^3 + 1)$ and $g(x) = (x^2 - 1)(x^2 - x + 1)$. If $h(x)$ is a polynomial such that $f(x) = g(x)h(x)$, then what is the value of $h(1)$?

10. The temperature on a summer evening can be determined by counting the number of chirps a cricket makes in 15 seconds and adding 38. The speed of a hypothetical ant varies with the temperature in such a way that the number of inches that the ant moves in 12 minutes is 38 less than the temperature. How many times does a cricket chirp while the ant travels 10 inches?

11. A girl goes up a ski lift at 4 miles per hour, and comes down the ski slope at 24 miles per hour. If the ski slope is the same length as the ski lift, and we ignore any time spent at the top, what is her average speed for the round trip, in miles per hour?

12. Seven white socks and four black socks are in a bag. Two socks are drawn at random, without replacement. What is the probability that they have the same color? Write as a reduced fraction.

13. A long escalator goes from a subway track to the street and moves up at constant rate. One person requires 40 seconds to walk up while the excalator is moving, and takes 40 steps in the process. Another person requires 50 seconds to walk up while the escalator is moving, and takes 20 steps in the process. How many steps of the escalator are required to go from the subway to the street if the escalator is not moving?

14. Suppose that the numbers 1, 2, 4, 8, 16, 32, 64, 128, and 256 are placed into the 9 squares in a 3-by-3 grid in such a way that each of the 9 numbers appears exactly once, and the product of the numbers appearing in any row or column is the same. What is the value of the product in each row and column?

15. Politician A lies on Mondays, Tuesdays, and Wednesdays and tells the truth on the other days of the week. Politician B lies on Thursdays, Fridays, and Saturdays, and tells the truth on the other days of the week. One day both of them say "Yesterday was one of my lying days." What day is it when they say this?

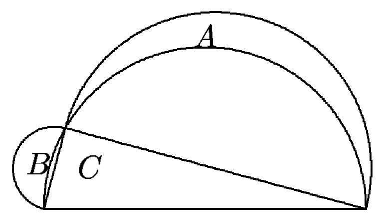
16. What is the smallest positive integer which cannot occur as the difference between two positive prime numbers?

17. Write, in interval form, the solution set of the inequality $|x^2 - 2x - 2| > |x^2 - 2x + 2|$.

18. What is the y-component of the center of the circle which passes through $(-1, 2)$, $(3, 2)$, and $(5, 4)$?

19. Write $(2001^3 - 1986^3 - 15^3)/(2001 \cdot 1986 \cdot 15)$ as an integer or reduced fraction.

20. Semicircles are drawn with each side of a right triangle as diameters, as in the figure. Let A, B, and C denote the indicated areas. Thus C is the area of the triangle, while A and B are the areas inside the smaller semicircles which lie outside the large semicircle. Express C in terms of A and B.



21. If $\sin\theta + \sin^2\theta + \sin^3\theta + \cdots = 4$, then what is the larger of the two possible values for $\cos\theta + \cos^2\theta + \cos^3\theta + \cdots$?

22. Concentric circles of radius 7 and 11 are drawn in a plane. A chord of the larger circle is trisected at its points of intersection with the smaller circle. What is the length of the chord?

23. In how many ways can you walk up a stairway which has 7 steps if you can take 1 or 2 steps at a time? (For example, you can walk up a stairway which has 3 steps in 3 different ways: 1-1-1, 1-2, or 2-1.)

24. What is the only positive integer which is divisible by 72 and has digits ascending in consecutive order? (Thus each digit must be 1 greater than its predecessor.)

25. A circle is inscribed in a square. In any one corner of the square, there is an isosceles right triangle which shares a vertex with the square and has hypotenuse tangent to the circle. What is the ratio of the area of this triangle to the area of the square?

26. Let L equal the largest of 2^{35}, 5^{15}, and 6^{14}, and S the smallest of these three numbers. Write the ordered pair (L, S), using the given exponential form for the two numbers that you write.

27. Beginning with 1, all the positive integers are written successively, beginning 1234567891011121314.... What digit appears in the 2001$^{\text{st}}$ position?

28. A boy who runs 8 miles per hour is 3/8 of the way across a railroad bridge when he hears a train whistle behind him. If he runs back, he will leave the bridge at precisely the moment the train enters it. If instead he keeps on running to the far end of the bridge, the train will reach him just as he leaves the bridge. What is the speed of the train in miles per hour?

29. What is the smallest positive integer $n > 150$ such that $\binom{n}{151}$ is divisible by $\binom{n}{150}$ but not equal to it?

30. Three vertices of a cube in space have coordinates $A(3,4,1)$, $B(5,2,9)$, and $C(1,6,5)$. What are the coordinates of the center of the cube?

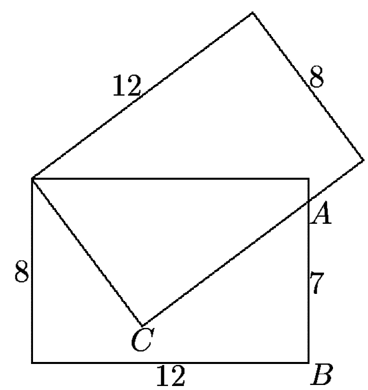
31. Let A and B denote the points of intersection of the circles $x^2+y^2-6x+4y=3$ and $x^2+y^2+4x-4y=17$. What is the slope of the segment AB?

32. Let $x_1 < x_2 < x_3 < \cdots$ be a listing of all the numbers which can be written as a sum of one or more distinct powers of 3. For example, $x_1 = 3^0 = 1$, $x_2 = 3$, and $x_3 = 4$. What is the value of x_{100}?

33. Find a polynomial of degree 4 with integer coefficients which has $\sqrt{2}+\sqrt{5}$ as a root.

34. A line segment parallel to an edge of a regular hexagon of side 1 divides that hexagon into two pieces whose ratio of areas is 1:2. What is the length of that line segment?

35. Let two 8×12 rectangles share a common corner and overlap as in the diagram below, so that the distance AB from the bottom right corner of one rectangle to the intersection point A along the right edge of that rectangle is 7. What is the area of the region common to the two rectangles?



36. If the lower left corner of the above diagram is at (0,0), what are the coordinates of the point C which is the lowest vertex of the slanted rectangle?

37. What is the value of the positive integer n for which the least common multiple of 36 and n is 500 greater than the greatest common divisor of 36 and n?

38. Suppose that the sum of the squares of two complex numbers w and z is 7, the sum of their cubes is 10, and their sum is a real number. What is the largest real value that $w+z$ can have?

39. You have a binary counter attached to a button. Each time you press the button, the counter increases by 1, expressing the number in binary notation. Each time a digit flips in the counter, you pay \$1. The counter is initially at 0. What is the cost of pressing the button 2001 times? (2001 in binary is 11111010001). For example, the cost of pressing the button 3 times is \$4, since it will go from 00 to 01 to 10 to 11, at a cost of \$1+\$2+\$1.

40. What positive number x satisfies $\sqrt[3]{x+9} - \sqrt[3]{x-9} = 3$.

SOLUTIONS TO 2001 EXAM

After the correct answer, in square brackets we list the number of people out of 159 who answered it correctly, followed by the number out of the 33 people who scored at least 16 who answered it correctly.

1. 6. [150,32] It is the reciprocal of $\frac{3}{6} - \frac{2}{6} = \frac{1}{6}$.
2. 5. [141.5, 33] $5n^3 = 5^4$. Thus $n = 5$.
3. 25. [98,32] The new price is 0.8 times the old price. Thus the amount that a dollar will purchase is $1/0.8 = 1.25$ times what it used to be.
4. [98,32] $\pi/2$. If the radius is r, then the inscribed square is composed of four isosceles right triangles with both legs equal to r. The ratio is $\pi r^2/(4r^2/2)$.
5. [81,29] 6. The impossible scores are 1, 2, 4, 5, 8, and 11. The scores that can be made with 0 touchdowns are 0, 3, 6, 9, ... The scores that can be made with 1 touchdown are 7, 10, 13, ... The scores that can be made with 2 touchdowns are 14, 17, ... Three or more touchdowns will duplicate one of these scores. It is easy to enumerate the missing scores.
6. 8. [116,33] Triangles CDE and CAB are similar. Thus CE:CD=EB:AD. Hence CE:4=2:1.
7. 0. [79,26] Both legs cannot be odd, since $(2a'+1)^2 + (2b'+1)^2 = 4A+2$, which is not a square. We must have $c^2 = a^2 + 2^2$ with c and a odd primes. Then $4 = c^2 - a^2 = (c-a)(c+a)$ with $c - a \geq 2$ and $c + a > 2$, which is impossible.
8. $\sqrt{10}$. [15,10] r^2 increased by 5 in a 75-year period. It will increase by 1, to 10, in another 15 years.
9. 5. [29,12] Factor $(x^5-1) = (x-1)(x^4+x^3+x^2+x+1)$ and $(x^2-1) = (x-1)(x+1)$ and cancel the $(x-1)$ factors in the equation. Obtain $5 \cdot 2 = 2 \cdot 1 \cdot h(1)$. Or get $h(x) = x^4 + x^3 + x^2 + x + 1$.
10. 480. [69,27] The inches moved in 12 minutes equals the chirps in 1/4 minute. Thus in any period of time, the number of chirps is 48 times the number of inches moved.
11. 48/7. [75.5,30] If the length of the slope is D miles, her speed is $2D/(\frac{D}{4} + \frac{D}{24}) = 2/\frac{7}{24} = 48/7$.
12. 27/55. [59.5,25.5] It is $(\binom{7}{2} + \binom{4}{2})/\binom{11}{2}$ or $\frac{7}{11} \cdot \frac{6}{10} + \frac{4}{11} \cdot \frac{3}{10}$. Either gives 54/110.
13. 120. [60,28] In the extra 10 seconds, the escalator moved 20 steps. Thus the escalator goes two steps per second, and so in the 40 seconds it went 80 steps. $80 + 40 = 120 = 100 + 20$.
14. 4096. [75.5,28] The exponents from 0 to 8 are being placed in the squares so that the sum in each row and column is the same. The common sum is $(1 + \cdots + 8)/3 = 12$. Thus the common product is $2^{12} = 4096$.

15. Thursday. [143,33] The only days that Politician A would make that statement are Monday and Thursday, and the only days that Politician B would make that statement are Thursday and Sunday.

16. 7. [52,25] The smaller numbers can be obtained as $3-2$, $5-3$, $5-2$, $7-3$, $7-2$, and $11-5$. The difference of odd primes is even, and since 9 is not prime, 7 cannot occur as an odd prime minus an even one.

17. $(0,2)$. [32,11] It will be the open bounded interval whose endpoints satisfy $x^2-2x+2 = -(x^2-2x-2)$, so $x^2-2x=0$.

18. 6. [57,19] We want y such that $(1,y)$ is equidistant from $(3,2)$ and $(5,4)$. Thus $2^2+(y-2)^2 = 4^2+(y-4)^2$. This reduces to $4y=24$.

19. 3. [15,9] With $a=1986$ and $b=15$, it is $((a+b)^3-a^3-b^3)/((a+b)ab)=3$.

20. $A+B$. [35,15] Let W denote the intermediate area, and let ℓ, w, and h denote the three sides of the triangle, with h the hypotenuse. Then $C+W=\frac{\pi}{8}h^2$, while $A+B+W=\frac{\pi}{8}\ell^2+\frac{\pi}{8}w^2$. By the Pythagorean Theorem, $C=A+B$.

21. 3/2. [7,7] $\sin\theta/(1-\sin\theta)=4$ and hence $\sin\theta=4/5$. Thus $\cos\theta=3/5$ and so $\cos\theta/(1-\cos\theta)=3/2$.

22. 18. [14,8] If the circles are centered at (0,0) and the chord is a (vertical) line of constant x, then $\sqrt{121-x^2}=3\sqrt{49-x^2}$. Then $x^2=40$, and so the length is $2\sqrt{121-40}$.

23. 21. [57.5,23] One way with all 1's; $\binom{6}{1}$ ways with one 2; $\binom{5}{2}$ ways with two 2's; and $\binom{4}{3}$ ways with three 2's. (In fact, as the number of steps increases, the number of ways that you can do this forms the Fibonacci sequence 2, 3, 5, 8, 13, 21, ...)

24. 3456. [46,21] The number formed by the last two digits must be divisible by 4. So it must end with 56. (12 is too small.) The sum of the digits must be divisible by 9. Working backwards from the 6 yields the result.

25. $(3-2\sqrt{2})/4$. [13,12] Let the side length of the square be 1. The altitude of the little triangle is $(\sqrt{2}-1)/2$. The area of an isosceles right triangle equals the square of its altitude. (This is true because if such a triangle has legs of length s, then its altitude is $s/\sqrt{2}$.) Thus the answer is $(\sqrt{2}-1)^2/4$.

26. $(6^{14},5^{15})$. [33,13] Since $32<36$, $(2^5)^7<(6^2)^7$. Since $125<128$, $(5^3)^5<(2^7)^5$.

27. 3. [48,22] The numbers through 99 require $9+180$ digits. Since $(2001-189)/3=604$, it will be the third digit of 703.

28. 32. [52,24] Let L denote the length of the bridge, D the distance from the train to the bridge when the boy hears the whistle, and S the speed of the train. Then $D/S=\frac{3}{8}L/8$ and $(D+L)/S=\frac{5}{8}L/8$. Thus $(D+L)/D=5/3$, so $D=\frac{3}{2}L$ and $S=32$. Alternatively, when the boy is 3/4 of the way through the tunnel, the train will just be entering it. Hence it must be going 4 times as fast as the boy in order to catch him at the far end.

29. 452. [6,5] $\binom{n}{151}/\binom{n}{150}=(n-150)/151$. We must have $n-150=302$.

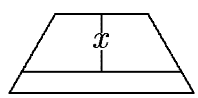
30. (4,3,5). [8,6] The lengths of the three segments are $AB = \sqrt{72}$, $AC = \sqrt{24}$, and $BC = \sqrt{48}$. Thus AB must be a diagonal of the cube, and its midpoint the center of the cube.

31. 5/4. [21.5,11.5] It is the negative reciprocal of the slope of the line connecting the centers. The centers are at $(3, -2)$ and $(-2, 2)$. The slope of the line connecting the centers is $-4/5$. Alternatively, subtract the two equations and find that any points lying on both must lie on the line $10x - 8y = 14$, which has slope 5/4.

32. 981. [3,3] Each positive integer can be written uniquely as a sum of distinct powers of 2. The ordering of the sums of distinct powers of 3 will be the same as the ordering of the sums of distinct powers of 2 with corresponding exponents. Thus x_n will have the same exponents of 3 as occur in the decomposition of n as a sum of distinct powers of 2. For example $x_3 = 3^1 + 3^0$ corresponds to $3 = 2^1 + 2^0$. Since $100 = 2^6 + 2^5 + 2^2$, $x_{100} = 3^6 + 3^5 + 3^2$.

33. $x^4 - 14x^2 + 9$. [2,0] Set $x = \sqrt{2} + \sqrt{5}$. Then $(x - \sqrt{2})^2 = 5$. Thus $x^2 - 3 = 2\sqrt{2}x$. Squaring both sides and expanding yields the desired result.

34. $\sqrt{3}$. [4,1] We draw the top half of the hexagon below. We wish to choose x so that the area above the line is 2/3 of the area of the trapezoid. The height of the trapezoid is $\frac{1}{2}\sqrt{3}$, and its bases are 1 and 2. Thus its area is $\frac{3}{4}\sqrt{3}$. We want x so that $x\frac{1}{2}(2+\frac{2x}{\sqrt{3}}) = \frac{1}{2}\sqrt{3}$. Then our desired length will be $1+\frac{2x}{\sqrt{3}}$. We have $2x^2+2x\sqrt{3}-3 = 0$ and hence $x = (3 - \sqrt{3})/2$. Thus the desired length is $\sqrt{3}$.



35. 42. [10,2] Let D denote the upper left vertex of the lower rectangle. The desired area is the sum of two right triangles with common hypotenuse AD. One has legs 1 and 12, so the hypotenuse is $\sqrt{145}$. The other has one leg 8, so the other leg must be $\sqrt{145 - 64} = 9$. Thus the area is $\frac{1}{2}(12 + 72)$.

36. $(4.8, 1.6)$. [0,0] Let D denote the vertex at which the two rectangles intersect. The tangent of the angle at D between the horizontal and downward slanting lines is

$$\tan(\arctan \tfrac{1}{12} + \arctan \tfrac{9}{8}) = \frac{\frac{1}{12} + \frac{9}{8}}{1 - \frac{1}{12}\frac{9}{8}} = \frac{4}{3}.$$

Thus the line DC goes over $\frac{3}{5}8$ and down $\frac{4}{5}8$.

37. 56. [6,6] The lcm must be even, hence the gcd must be even. The only possible even values for the gcd are 36, 18, 12, 6, 4, and 2. When 500 is added to these, the only sum divisible by 36 is $504 = 2^3 3^2 7$. Since $\gcd(2^2 3^2, n) = 2^2$ and $\text{lcm}(2^2 3^2, n) = 2^3 3^2 7$, we must have $n = 2^3 7$.

38. 4. [8,1] In order that $w + z$ and $w^2 + z^2$ are both real, we must have $w = x + iy$ and $z = x - iy$. Then $x^2 - y^2 = \frac{7}{2}$ and $x^3 - 3xy^2 = 5$. This yields the equation $x(21 - 4x^2) = 10$ or $0 = 4x^3 - 21x + 10$. This equation has one real solution greater than 1. Try $x = 2$ and find that it works. The desired sum is $2x = 4$. The values of z and w are $2 \pm \frac{i}{\sqrt{2}}$.

39. \$3995. [6.5,4] The 1's digit will change every time, the 2's digit every 2nd time, the 4's digit every 4th time, etc. The number of dollars paid due to changes in the 1's digit has binary expansion 11111010001, the number of dollars paid due to changes in the 2's digit has binary expansion 1111101000, the number of dollars paid due to changes in the 4's digit has binary expansion 111110100, etc. Thus the total number of dollars paid will be the number corresponding to the sum of binary numbers beginning with 11111010001 plus this number with any number of digits cut off the right side. Each 1 which appears in the dth position in 11111010001 will also appear once in each position to the right of the dth in one of the following numbers in the sum. Thus, for example, the initial 1 will appear once in each position from the 2^0 position up to the 2^{10} position. The number containing such 1's in its binary expansion is $2^{11} - 1$. The sum for all the 1's is $(2^{11}-1)+(2^{10}-1)+(2^9-1)+(2^8-1)+(2^7-1)+(2^5-1)+(2^1-1)$. The sum of the 2-powers here is $2 \cdot 2001$ and 7 is subtracted from it.

40. $\sqrt{80}$. [1,1] Cube both sides to get $x + 9 - 3(x + 9)^{2/3}(x - 9)^{1/3} + 3(x + 9)^{1/3}(x - 9)^{2/3} - (x - 9) = 27$. This simplifies to $(x^2 - 81)^{1/3}((x - 9)^{1/3} - (x + 9)^{1/3}) = 3$. By the original equation, the expression in parentheses equals -3. Thus the equation reduces to $x^2 - 81 = -1$.