2019 LEHIGH UNIVERSITY HIGH SCHOOL MATH CONTEST

1. Write as a simple reduced fraction $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$.
2. The hypotenuse of a right triangle is 39, and the ratio of its legs is 5:12. What is its area?
3. Every minute, from the first to the thirtieth, you remove some coins from a jar. If the number of the minute is odd, remove 3, and if the number of the minute is even, remove n, where n is the number of the minute. For example, during the first four minutes, you remove 3, 2, 3, 4. After the minute-30 removal, there are 10 coins remaining. How many were in the jar initially?
4. Sally runs one mile at 6 miles per hour and then the next mile at 8 miles per hour. What is her average speed in miles per hour for the 2-mile run?
5. What is the equation of the circle having the points $(-3, 5)$ and $(5, -1)$ as endpoints of a diameter? Any correct form (involving x and y) is acceptable.
6. Let $s(n)$ denote the sum of the digits in n. Let $s^2(n) = s(s(n))$, $s^3(n) = s(s(s(n)))$, etc. What is the value of $s^{2019}(2019)$?
7. Five bricks are weighed two-at-a-time in all possible ways. The weights obtained are 60, 62, 63, 64, 65, 66, 67, 68, 70, and 71. What is the difference between the heaviest weight and the lightest weight?
8. For how many positive integers y does there exist a positive integer x such that $x^2 - y^2 = 72$?
9. A bus holds up to 45 people. It doesn't have to be full. Adults pay \$8, and children pay \$5. If the total paid by all riders is exactly \$250, what is the maximum possible number of children on the bus?

10. Find the ordered pair (a, b) of positive integers with $a < b$ such that $a^2 + b^2 + a^2b^2 = 2004$.

11. In how many ways can four indistinguishable coins be placed on a 5-by-5 grid so that no three of them lie on the same diagonal? Here we would be considering diagonals of length 3, 4, or 5.

12. Two bikers are 3/4 of the way through a mile-long tunnel when a train traveling 40 mph approaches the closer end of the tunnel. (It is still some distance away from the tunnel.) The bikers decide to go in opposite directions, and each escapes the tunnel just as the front of the train gets to them. They bike at the same speed. What is that speed (in mph)?

13. A sphere sits in a corner of a rectangular room, touching two adjacent walls and the floor. A point on the sphere is 5 units from each of the walls that it touches and 10 units from the floor. List all possible values of the radius of the sphere.

14. What is the smallest positive solution (in radians) of the following equation?

$$5 \cos x + 2 \sin^2 x = 4$$

15. What is the largest value of $3a + 5b$ taken over all ordered pairs (a, b) of nonnegative integers for which $2^a 3^b < 5000$?

16. A function f is defined on the positive integers satisfying that, for each n, the sum of $f(d)$, as d ranges over all positive divisors of n, equals n. What is $f(20)$?

17. Let P be the (filled-in) parallelogram with vertices $(1, 0)$, $(2, 0)$, $(1, 4)$, and $(0, 4)$. Let Q be the (filled-in) parallelogram with vertices $(0, 2)$, $(0, 3)$, $(4, 4)$, and $(4, 3)$. What is the area of the intersection of P and Q?

18. What is the smallest integer > 1 which is at least 600 times as large as any of its prime divisors?

19. For how many positive integers n less than 363636 is n divisible by $\lfloor\sqrt{n}\rfloor$?

20. What is the maximum area of a quadrilateral with sides 4, 8, 11, and 13, in some order?

21. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2(n+2)^2}$.

22. Let $ABCD$ be a trapezoid with AD parallel to BC, $AD = 3$, and $BC = 1$. Let P on AD satisfy $AP = 2$. Let Q on CD be such that BD, CP, and AQ pass through the same point. What is the ratio $CQ : QD$?

23. Let A and B be randomly chosen subsets of $\{1, 2, 3, 4, 5, 6\}$, possibly equal. This means that every subset, including the empty set and the whole set, is equally likely to be selected. What is the probability that one of the subsets contains the other? This includes the case in which they are equal, and allows either one to be the containing set.

24. Numbers a_1, a_2, a_3 are in arithmetic progression with $a_1 + a_2 + a_3 = 45$. Numbers g_1, g_2, g_3 are in geometric progression. These satisfy $a_1 + g_1 = 32$, $a_2 + g_2 = 29$, and $a_3 + g_3 = 33$. What is the smallest number that can possibly occur in either sequence?

25. Let D be the midpoint of BC in triangle ABC. If $\angle BAD = 30$ degrees and $\angle DAC = 15$ degrees, how many degrees are in $\angle ABC$?

26. A function $f(m, n)$ satisfies $f(1, n) = n$, $f(n, 1) = 1$, and $f(m, n) = f(m-1, n)f(m, n-1)$ for $m, n \geq 2$. What is the largest integer k such that 3^k divides $f(9, 9)$?

27. Hexagon $ABCDEF$ is inscribed in a circle. $AB = BC = CD = 1$ and $DE = EF = FA = 2$. What is the radius of the circle?

28. Find the number of ordered triples (a_1, a_2, a_3) of distinct integers with $|a_i| \leq 10$ for $i = 1, 2, 3$ and $a_1 + a_2 + a_3 > 0$.

29. Let $x = \underline{a}\,\underline{b}\,\underline{c}$ be the integer formed by writing the digits of positive integers a, b, and c in order, where b and c are 2-digit numbers. What is the smallest value of x such that $x = (a + b + c)^3$?

30. Find all numbers of the form p^n, where p is a (positive) prime number and n is a positive integer, which can be obtained as $2x^2 + 3x - 2$ for some integer x.

31. If x, y, and z are positive real numbers satisfying $x^2 + y^2 = 9$, $y^2 + yz + z^2 = 16$, and $x^2 + \sqrt{3}xz + z^2 = 25$, what is the value of $2xy + xz + \sqrt{3}yz$?

32. Equilateral triangle ABC with side length 4 is inscribed in a circle. Point P on minor arc AC satisfies $AP \cdot PC = 5$. What is the length of BP?

33. What is the largest integer for which every digit, except the first and last, is strictly less than the arithmetic mean of its two neighboring digits?

34. List all values of c such that, if $f(x) = x^2 + 4x + c$, then $f(f(x))$ has exactly three distinct real roots.

35. What is the size of the largest set of non-negative integers such that $|m - n| \geq (m+1)(n+1)/20$ for all m, n in the set with $m \neq n$?

36. Let v and w be randomly selected distinct solutions of the complex equation $z^{2019} - 1 = 0$. What is the probability that $|v + w| \geq \sqrt{2 + \sqrt{2}}$?

37. Each side of a square contains one of the points $(0, 9)$, $(8, 5)$, $(6, 0)$, and $(-2, 4)$. What are the possible values of its area? (The specified points are not vertices of the square.)

38. Positive real numbers $x_1, \ldots, x_{100}$ satisfy $\sum_{i=1}^{100} x_i = 101 = \sum_{i=1}^{100} \frac{1}{x_i}$. What is the largest value of $x_i + \frac{1}{x_i}$ that can possibly occur?

39. Let S denote the set of ordered pairs (m, n) of positive integers, both of which are ≤ 100. Say that elements $P_1 = (m_1, n_1)$ and $P_2 = (m_2, n_2)$ in S are *closely related* if $m_1 n_1 = m_2 n_2$ or $m_1/n_1 = m_2/n_2$. Then say that elements P and Q of S are *related* if there is a sequence $A_1, A_2, \ldots, A_n$ with $A_1 = P$ and $A_n = Q$ such that A_i and A_{i+1} are closely related for $1 \leq i \leq n - 1$. How many elements of S are related to $(1, 1)$?

40. Let triangle ABC and point X_1 on AC satisfy $AB = AC = 91$, $BC = 70$, and $X_1C = 65$. Let $X_0 = B$. Points $X_2, X_4, X_6, \ldots$ on AB and $X_3, X_5, X_7, \ldots$ on AC satisfy $X_{n+1}X_n \perp X_nX_{n-1}$ for $n \geq 1$. Find $\sum_{n=1}^{\infty} X_{n-1}X_n$.

SOLUTIONS, annotated with the number of people answering correctly, out of the 54 people who scored at least 18.

1. 13/12 (or $1\frac{1}{12}$). [54] It is $(6+4+3)/12$.
2. 270. [52] It is a multiple of a 5-12-13 right triangle, so its legs are 15 and 36, and its area is $15 \cdot 18$.
3. 295. [51] It is $10+15\cdot3+(2+4+\cdots+28+30) = 55+7\cdot32+16 = 295$. Here we have considered the sum as seven pairs summing to 32, plus the 16 in the middle.
4. 48/7 or $6\frac{6}{7}$. [52] It is $2/(\frac{1}{6}+\frac{1}{8}) = 48/7$.
5. $(x-1)^2+(y-2)^2 = 25$ or $x^2-2x+y^2-4y = 20$. [52] The center is halfway between the two points, and the diameter is $\sqrt{8^2+6^2}$.
6. 3. [54] $s(2019) = 12$ so $s^k(2019) = 3$ for all $k \geq 2$.
7. 8. [48] Since there are $\binom{5}{2} = 10$ pairs of bricks and 10 distinct sums, all bricks must have distinct weights. Let $a < b < c < d < e$ be the weights. It must be that $a+c$ is the second smallest sum, and $c+e$ the second largest, so $e-a = 70-62 = 8$. (The weights are 29, 31, 33, 34, and 37.)
8. 3. [50] We must have $(x-y)(x+y)$ a factorization $a \cdot b$ of 72 with both factors of the same parity so that $x = \frac{a+b}{2}$ is an integer. Such factorizations are $2 \cdot 36$, $4 \cdot 18$, and $6 \cdot 12$. The pairs (x, y) are $(19, 17)$, $(11, 7)$, and $(9, 3)$.
9. 34. [47] If A is the number of adults and C the number of children, then $8A + 5C = 250$ and $A + C \leq 45$. A must be a multiple of 5, and we want it to be as small as possible. If $A = 0$ or 5, then $C = 50$ or 42. In either case $A + C > 45$. If $A = 10$, then $C = 34$ and $A + C = 44$.
10. $(2, 20)$. [54] We have $(a^2+1)(b^2+1) = 5 \cdot 401$. So clearly $a = 2$, $b = 20$ works. It is easy to see that nothing else does.

11. 11812. [18.5] There are $\binom{25}{4} = 12650$ ways to place the coins on the grid. We subtract from this the $4\binom{4}{4} + 2\binom{5}{4} = 14$ ways to have all four on a diagonal and the $4\binom{3}{3}22 + 4\binom{4}{3}21 + 2\binom{5}{3}20 = 824$ ways to have exactly three on a diagonal. For example, the $4\binom{4}{3}21$ is because there are four length-4 diagonals, with $\binom{4}{3}$ ways to place three coins on each, and 21 choices for the position of the other coin.

12. 20 (mph). [51] If the initial distance from the train to the tunnel is x miles, and the riders' speed is s, then $\frac{1}{4}/s = \frac{x}{40}$ and $\frac{3}{4}/s = \frac{x+1}{40}$. Subtracting these yields $\frac{1}{2}/s = \frac{1}{40}$ and so $s = 20$. **Another solution:** When the train reaches the first end of the tunnel, both bikers have traveled $\frac{1}{4}$ mile, so the second biker is at the center of the tunnel. He then bikes the remaining $\frac{1}{2}$ mile in the same time it takes the train to traverse the entire 1-mile long tunnel, so he is going half as fast as the train.

13. 5 and 15. [34] If the corner of the room is $(0, 0, 0)$, the equation of the sphere is $(x-r)^2 + (y-r)^2 + (z-r)^2 = r^2$. Substituting $x = y = 5$ and $z = 10$ yields $2r^2 - 40r + 150 = 0$, whose solutions are $r = 5$ and 15.

14. $\pi/3$. [51] $5\cos x + 2(1 - \cos^2 x) = 4$ simplifies to $0 = (2\cos x - 1)(\cos x - 2)$. $\cos x = \frac{1}{2}$ when $x = \pi/3$.

15. 38. [35] For $a = 0, 1, 2, 3, 4$, the largest b for which $2^a 3^b < 5000$ is $7, 7, 6, 5, 5$. The successive values of $3a + 5b$ are $35, 38, 36, 34, 37$. If $a \geq 5$, $2^{a-5}3^{b+3} < 2^a 3^b$ and gives the same value of the desired function, so there is no need to consider values of $a > 4$.

16. 8. [51] It equals $20 - f(1) - f(2) - f(4) - f(5) - f(10) = 20 - 1 - 1 - 2 - 4 - 4$.

17. 16/17. [42] The figure is a rectangle because its sides have slopes $1/4$ and -4, and by symmetry it is a square. Adjacent vertices of the square occur at the intersection of the line $y = 4 - 4x$ with the lines $y = 2 + \frac{1}{4}x$ and $y = 3 + \frac{1}{4}x$. These are the points $(\frac{8}{17}, \frac{36}{17})$ and $(\frac{4}{17}, \frac{52}{17})$. The area of the square is $(4^2 + 16^2)/17^2 = 16/17$.

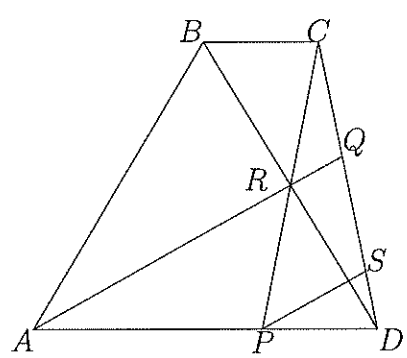
18. 1944. [33] If the largest prime divisor is 2, the smallest 2-power ≥ 1200 is 2048. If the largest prime divisor is 3, then the numbers that we consider (smallest product greater than 1800) are $3 \cdot 1024$, $9 \cdot 256$, $27 \cdot 128$, $81 \cdot 32$, $243 \cdot 8 = 1944$, $729 \cdot 4$, and 2179. If the largest prime divisor is greater than 3, then the number must be at least $600 \cdot 5$.

19. 1807. [42] If $\lfloor \sqrt{n} \rfloor = m$, then $m^2 \leq n < (m+1)^2$, so $n = m^2$, $m^2 + m$, and $m^2 + 2m$ are the numbers divisible by m. Since $603^2 = 363609$, there are $3 \cdot 602 + 1$ values of n.

20. 70. [26] If the sides of length 4 and 13 are not adjacent, then draw a diagonal and flip one of the triangles, which makes the 4 and 13 adjacent without changing the area. Since $4^2 + 13^2 = 185 = 8^2 + 11^2$, we can form a quadrilateral consisting of two right triangles with common hypotenuse $\sqrt{185}$ as diagonal. Its area is $\frac{1}{2}(4 \cdot 13 + 8 \cdot 11) = 70$. The formula $A = \frac{1}{2}\sin(\alpha)\ell_1\ell_2$ implies that no larger area can be obtained.

21. 5/16. [38] If S is the desired sum, then $\sum\limits_{n\geq 1} \frac{1}{n^2} = \sum\limits_{n \geq 1} \frac{n^2+4n+4}{n^2(n+2)^2} = \sum\limits_{n\geq 1} \frac{1}{(n+2)^2} + 4S$. Therefore $1 + \frac{1}{2^2} = 4S$.

22. 2/3 or 2:3. [49] Let R be the intersection point. Triangles BCR and DPR are congruent. Let S on QD be such that $PS \parallel AQ$. Similar triangles imply that $QS = 2\,SD$ and that $CQ = QS$, from which the result follows.



23. 697/2048. [27] $A \subseteq B$ if and only if, for none of the elements x, $x \in A$ but $x \notin B$. Thus the probability that $A \subseteq B$ is $(\frac{3}{4})^6$. The desired probability is twice this minus the probability that $A = B$, since that possibility has been counted twice. The probability that $A = B$ is $(\frac{1}{2})^6$. Thus the answer is $(2 \cdot 3^6 - 2^6)/4^6 = (729 - 32)/2048$.

24. 4. [39] Write the sequences as $15 - d, 15, 15 + d$ and b, br, br^2. Then $-d + b = 17$, $br = 14$, and $d + br^2 = 18$. Adding the first and third, with the second used to replace b, we obtain $\frac{14}{r} + 14r = 35$, from which we obtain $r = 2$ or $\frac{1}{2}$. With $r = 2$, the sequences are 25, 15, 5 and 7, 14, 28. With $r = \frac{1}{2}$, they are 4, 15, 26 and 28, 14, 7.

25. 105. [18] By the Law of Sines in triangles ACD and ABD, we obtain $\frac{\sin(C)}{\sin(15)} = \frac{\sin(135-C)}{\sin(30)}$, and hence $\sin(C) = \frac{1}{2\cos(15)} \cdot \frac{\sqrt{2}}{2}(\cos(C)+\sin(C))$. Since $\cos(15) = \frac{\sqrt{3}+1}{2\sqrt{2}}$, we obtain $\sqrt{3}+1 = \cot(C) + 1$. Thus $\angle C = 30$, and so $\angle B = 180 - 45 - 30$.

26. 1838. [11] Let $P(m, n)$ denote the largest k such that 3^k divides $f(m, n)$. Then $P(1, 3) = 1$, $P(1, 6) = 1$, $P(1, 9) = 2$, $P(1, n) = 0$ for other values of $n < 10$, $P(m, 1) = 0$, and $P(m, n) = P(m-1, n) + P(m, n-1)$. We claim that for $m, n \leq 9$,

$$P(m, n) = \binom{m+n-5}{m-2} + \binom{m+n-8}{m-2} + 2\binom{m+n-11}{m-2},$$

so $P(9, 9) = \binom{13}{7} + \binom{10}{7} + 2\binom{7}{7} = 1716 + 120 + 2$. It is like a sum of Pascal's triangles coming down from the three nonzero values of $P(1, n)$. Checking values when $m = 2$ shows that it works as claimed. **Alernatively:** you can fill out a table of values of P rather quickly from the recursive formula.

27. $\frac{1}{3}\sqrt{21}$. [21.5] Let O be the center of the circle, and $2\theta = \angle AOB$ and $2\phi = \angle ODE$. Then $6\theta + 6\phi = 360$, so $\theta + \phi = 60$. If r is the desired radius, then $\sin\theta = \frac{1}{2r}$ and $\sin\phi = \frac{1}{r}$. Considering $\sin(\theta + \phi)$, we obtain $\frac{\sqrt{3}}{2} = \frac{1}{r}\sqrt{1 - 1/(4r^2)} + \frac{1}{2r}\sqrt{1 - 1/r^2}$. We rewrite this as $\sqrt{3}r^2 = \sqrt{4r^2 - 1} + \sqrt{r^2 - 1}$. After squaring, manipulating, and squaring again, we obtain $(r^2 - 1)^2(3r^2 -$

$2)^2 = 4(4r^2-1)(r^2-1)$, which, after cancelling (r^2-1), simplifies to $r^4(9r^2-21) = 0$.

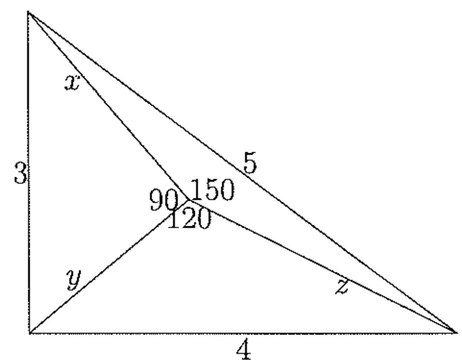
28. 3840. [3] The total number of all (a_1, a_2, a_3) with $|a_i| \le 10$ is $21 \cdot 20 \cdot 19 = 20^3 - 20$. We will subtract from that the number with $a_1 + a_2 + a_3 = 0$ and then divide by 2. The number with $a_1 < a_2 < a_3$ and $a_1+a_2+a_3 = 0$ is, for a_2 from -4 to 4, equal to 2, 4, 6, 8, 10, 8, 6, 4, 2, so a total of 50. Multiply by 6 to account for any order. Thus the desired answer is $4000 - 10 - 150$.

29. 91125. [3] Let $y = a + b + c$. Since 21^3 has only four digits, $y \ge 22$. Since $y^3 = 10^4a + 10^2b + c$, $y(y-1)(y+1) = y^3 - y = 9999a + 99b$ is divisible by 9 and by 11. Thus one of y, $y - 1$, and $y + 1$ is divisible by 11 and one is divisible by 9 (since at most one is divisible by 3). The smallest integers $y \ge 22$ having these properties are $y = 44$ and 45. Since $44^3 = 85184$ and $8 + 51 + 84 \ne 44$, 44 doesn't work. But $45^3 = 91125$ and $9 + 11 + 25 = 45$, so 45 works.

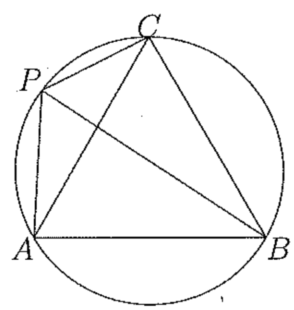
30. 3, 7, 25. [15] Write it as $(2x-1)(x+2)$. If either factor is ± 1, we obtain values -2, 3, -3, and 7 for the product. Otherwise both factors are multiples of p, and hence so is $2(x+2)-(2x-1) = 5$. Thus p must equal 5. If $x + 2 = 5^k$, then $2x - 1 = 2 \cdot 5^k - 5$, so $k = 1$, $x = 3$, and $2x^2 + 3x - 2 = 25$. If $2x - 1 = 5^k$, then $x + 2 = \frac{1}{2}(5^k + 5)$, so again $k = 1$ and $x = 3$.

31. 24. [17] By the Law of Cosines, x, y, and z are the lengths in the diagram below. The total area, which is 6, equals $\frac{1}{2}(xy +$

$\frac{1}{2}xz + \frac{\sqrt{3}}{2}yz)$.



32. $\sqrt{21}$. [21] Ptolemy's Theorem says $4 \cdot PC + 4 \cdot AP = 4 \cdot BP$. Thus $BP^2 = (PC + AP)^2 = PC^2 + AP^2 + 10$. By the Law of Cosines, $16 = AP^2 + PC^2 - 2AP \cdot PC \cos(120) = AP^2 + PC^2 + 5$. So $BP^2 = 16 - 5 + 10$.



33. 96433469. [16] Let $a_1, \ldots, a_r$ be the digits. If $x = a_{i-1} < a_i$, then $a_i \geq x + 1$, $a_{i+1} \geq x + 3$, $a_{i+2} \geq x + 6$, and a_{i+3} can't exist. So there can be at most two more digits after an increase. Similarly, if $x = a_i < a_{i-1}$, then $a_{i-1} \geq x + 1$, $a_{i-2} \geq x + 3$, $a_{i-3} \geq x + 6$, and a_{i-4} can't exist. So there can be at most two digits preceding a decrease. Thus the number can have at most 8 digits, and then the two middle digits must be repeated. The above inequalities show that the claimed number is the largest.

34. 0. [9] If f has only one distinct root, then $f \circ f$ has at most two. Thus f has two distinct roots. Let r be one for which $f(x) = r$ has only one solution. Then $x^2 + 4x + c - r = (x+2)^2$, so $c - 4$ is a root of f. Thus $(c-4)^2 + 4(c-4) + c = 0$, and so $c = 0$ or 3. For $c = 0$, we must have $x^2 + 4x = 0$ or -4, giving $x = 0, -4$, and -2. For $c = 3$, a root of $f \circ f$ must satisfy $x^2 + 4x + 3 = -3$ or -1, the two roots of f. Since $16 - 4 \cdot 6 < 0$, $f \circ f$ has at most two distinct real roots when $c = 3$.

35. 8. [9] Let $a = m+1$ and $b = n+1$ with $a > b > 0$. Let $k = a - b$. The condition becomes $k \geq \frac{b^2}{20-b}$ if $b < 20$, with no solutions if $b \geq 20$. For $b \leq 4$, this says $k \geq 1$. For values of b from 5 to 11, we have $k \geq 2, 3, 4, 6, 8, 10, 14$, respectively. A largest set of a'a and b's is $\{1, 2, 3, 4, 5, 7, 11, 25\}$. Decreasing these by 1 gives a maximal set of m's and n's.

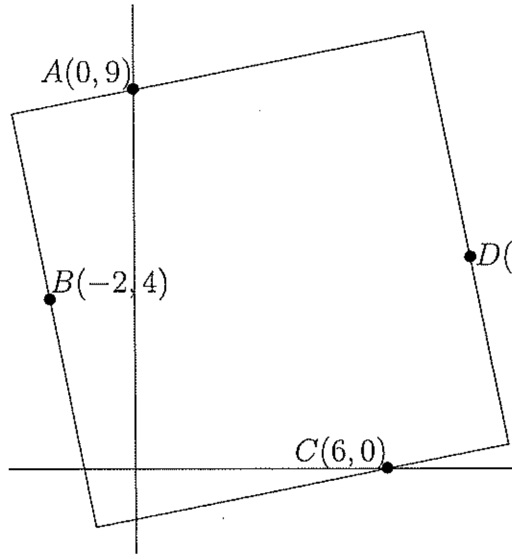
36. 252/1009. [7] Because of the symmetry of the solutions of the equation, we may assume $w = 1$. The possible v's are $\cos(2\pi k/2019) + i\sin(2\pi k/2019)$ for $k = \pm 1, \ldots, \pm 1009$. Then

$$\begin{aligned} & |v + w|^2 \\ = \; & \cos^2(2\pi k/2019) + 2\cos(2\pi k/2019) + 1 + \sin^2(2\pi k/2019) \\ = \; & 2 + 2\cos(2\pi k/2019), \end{aligned}$$

so our condition is $\cos(2\pi k/2019) \geq \sqrt{2}/2$. This occurs when $|2\pi k/2019| \leq \pi/4$, so $|k| \leq 252$. The desired probability is 252/1009.

37. 2601/26. [0] The lines through $A = (0, 9)$ and $C = (6, 0)$ are opposite sides of the square. Let $ax + by = 9b$ be the equation of the side passing through A. The perpendicular distance from C to that line is $|6a + 0b - 9b|/\sqrt{a^2 + b^2}$. The equation of the side of the square passing through $B = (-2, 4)$ is $-bx + ay = 2b + 4a$, and the distance from $(8, 5)$ to it is $|-8b + 5a - 2b - 4a|/\sqrt{a^2 + b^2}$. These distances must be equal so $6a - 9b = \pm(a - 10b)$. Thus either $b = -5a$ or $7a = 19b$. For the first, we can take $a = -1$, $b = 5$. The distance between parallel sides equals $51/\sqrt{26}$. The

other solution ($7a = 19b$) does not lead to a square since the line through A would separate B from C.



38. 401/101. [6] We may assume that the maximum occurs when $i = 100$. Thus $(101 - x_{100})(101 - \frac{1}{x_{100}}) = \sum_{i=1}^{99} x_i \sum_{i=1}^{99} \frac{1}{x_i} \geq 99^2$, by the Cauchy-Schwarz Inequality, with equality occurring if and only if all x_i are equal ($i \leq 99$). Thus $101^2+1-101(x_{100}+\frac{1}{x_{100}}) \geq 99^2$, so $x_{100}+\frac{1}{x_{100}} \leq \frac{401}{101}$. If we choose x_{100} so that equality occurs here, and then choose $x_1 = \cdots = x_{99} = \frac{1}{99}(101 - x_{100})$, then since $101 - x_{100} = \sum_{i=1}^{99} x_i$, it follows that $101 - \frac{1}{x_{100}} = \sum_{i=1}^{99} \frac{1}{x_i}$.

Approximating $\frac{401}{101}$ by 4 leads to $x_{100} \approx 2 + \sqrt{3} \approx 3.73$, and $x_1 = \cdots = x_{99} \approx 1 - \frac{1}{99}\sqrt{3} \approx .9825$. The actual values are $x_{100} = (401 + 3\sqrt{13333})/202 \approx 3.70$ and $x_i = (6667 - \sqrt{13333})/6666 \approx .9828$.

39. 310. [2] Note that, for $P_1 = (m_1, n_1)$ and $P_2 = (m_2, n_2)$, if $m_1 n_1$ is a perfect square and P_2 is closely related to P_1, then $m_2 n_2$ is a perfect square. This is true since if $m_2 n_2 \neq m_1 n_1$, then $m_2 n_2 = \frac{m_2}{n_2} \cdot n_2^2 = \frac{m_1}{n_1} \cdot n_2^2 = m_1 n_1 \cdot \frac{1}{n_1^2} \cdot n_2^2$. Thus the same is true for two elements which are related. Thus all elements related to $(1,1)$ must be of the form (ka^2, kb^2) with a and b relatively prime.

To see that all such elements are related to $(1,1)$, note that (ka^2, kb^2) is closely related to (kab, kab), which is closely related to $(1,1)$. If $\max(a,b) = c > 1$, there are $2\phi(c)$ pairs (a^2, b^2) and $\lfloor 100/c^2 \rfloor$ values of k. Here $\phi(c)$ is the number of positive integers $< c$ relatively prime to c. If $c = 1$, there is just one pair (a^2, b^2). Thus the desired answer is $100 + \sum_{c=2}^{10} 2\phi(c)\lfloor 100/c^2 \rfloor =$ $100+2\cdot 25+4\cdot 11+4\cdot 6+8\cdot 4+4\cdot 2+12\cdot 2+8\cdot 1+12\cdot 1+8\cdot 1$.

40. $553/5$ or 110.6. [3] Let M and N on BC be the bottom of perpendiculars from A and X_1. Then $\frac{NC}{MC} = \frac{X_1C}{AC}$ so $NC = \frac{65}{91}\cdot 35 = 25$. Next, $X_1N = \sqrt{65^2 - 25^2} = 60$ and $X_0X_1 = \sqrt{60^2 + 45^2} = 75$. Now $\tan(X_1BA) = \tan(ABC - X_1BC) = (\frac{12}{5} - \frac{4}{3})/(1 + \frac{12}{5}\frac{4}{3}) = \frac{16}{63}$. Hence $X_1X_2 = X_0X_1 \cdot \frac{16}{63} = \frac{400}{21}$, and $X_0X_2 = \sqrt{75^2 + (400/21)^2} = 25 \cdot 65/21$. Since $X_{n-1}X_n \parallel X_{n+1}X_{n+2}$, the ratio of similar triangles $AX_{n+1}X_{n+2}$ and $AX_{n-1}X_n$ equals AX_{n+2}/AX_n, as does the ratio of similar triangles $AX_{n+2}X_{n+3}$ to AX_nX_{n+1}. Thus this ratio r is the same for all n, and hence equals $AX_2/AX_0 = (91-\frac{25\cdot 65}{21})/91 = 1-\frac{125}{147}$. The desired sum is $(X_0X_1+X_1X_2)/(1-r) = (75+\frac{400}{21})\frac{147}{125} = (3\cdot 21+16)\frac{7}{5} = 553/5$.