HOMOTOPY TYPE AND v_1-PERIODIC HOMOTOPY GROUPS OF p-COMPACT GROUPS

DONALD M. DAVIS

Abstract. We determine the v_1-periodic homotopy groups of all irreducible p-compact groups. In the most difficult, modular, cases, we follow a direct path from their associated invariant polynomials to these homotopy groups. We show that, with several exceptions, every irreducible p-compact group is a product of explicit spherically-resolved spaces which occur also as factors of p-completed Lie groups.

1. Introduction

In [4] and [3], the classification of irreducible p-compact groups was completed. This family of spaces extends the family of (p-completions of) compact simple Lie groups. The v_1-periodic homotopy groups of any space X, denoted $v_1^{-1}\pi_*(X)_{(p)}$, are a localization of the portion of the homotopy groups detected by K-theory; they were defined in [20]. In [17] and [16], the author completed the determination of the v_1-periodic homotopy groups of all compact simple Lie groups. Here we do the same for all the remaining irreducible p-compact groups.[1]

Recall that a p-compact group ([22]) is a pair (BX, X) such that BX is p-complete and $X = \Omega BX$ with $H^*(X;\mathbb{F}_p)$ finite. Thus BX determines X and contains more structure than does X. The homotopy type and homotopy groups of X do not take into account this extra structure nor the group structure on X.

Date: September 21, 2007.

2000 *Mathematics Subject Classification.* 55Q52, 57T20, 55N15.

Key words and phrases. v_1-periodic homotopy groups, p-compact groups, Adams operations, K-theory, invariant theory.

The author is grateful to Kasper Andersen and Clarence Wilkerson for valuable suggestions related to this project.

[1]If the groups $v_1^{-1}\pi_i(X)$ are finite, then p-completion induces an isomorphism $v_1^{-1}\pi_*(X) \to v_1^{-1}\pi_*(X_p)$. ([9, p.1252])

According to [4, 1.1,11.1] and [3], the irreducible p-compact groups correspond to compact simple Lie groups[2] and the p-adic reflection groups listed in [2, Table 1] for which the character field is strictly larger than $\mathbb{Q}$. See [13, pp.430-431] and [25, p.165] for other listings. We use the usual notation $((BX_n)_p,(X_n)_p)$, where n is the Shephard-Todd numbering ([33] or any of the previously-mentioned tables) and p is the prime associated to the completion.

We will divide our discussion into four families of cases:

(1) The compact simple Lie groups—infinite family 1, part of infinite family 2, and cases 28, 35, 36, 37 in the Shephard-Todd list.
(2) The rest of the infinite families numbered 2a, 2b, and 3.
(3) The nonmodular special cases, in which p does not divide the order of the reflection group. This is cases 4-27 and 29-34.
(4) The modular cases, in which p divides the order of the reflection group. These are cases $(X_{12})_3$, $(X_{24})_2$, $(X_{29})_5$, $(X_{31})_5$, and $(X_{34})_7$. (Actually, we include $(X_{12})_3$ in Case (3) along with the nonmodular cases, and the Dwyer-Wilkerson space $(X_{24})_2$ was handled in [6].)

Here is a brief summary of what we accomplish in each case. The author feels that his contributions here are nil in case (1)[3], minuscule in case (2), modest in case (3), and significant in case (4).

(1) Spaces X_1, X_{28}, X_{35}, X_{36}, and X_{37} are, respectively, $SU(n)$, F_4, E_6, E_7, and E_8. These are p-compact groups for all primes p, although for small primes they were excluded by Clark and Ewing ([13]) because $H^*(BX;\mathbb{F}_p)$ is not a polynomial algebra. The exceptional Lie group G_2 is the case $m=6$ in infinite family 2b. The spaces $SO(n)$, $\operatorname{Spin}(n)$, and $Sp(n)$ appear in the infinite family 2a with $m=2$. Simplification of the homotopy types of many of these, when p is odd, to products of spheres and

[2]Cases in which distinct compact Lie groups give rise to equivalent p-compact groups are discussed in [4, 11.4].

[3]But he accomplished much in these cases in earlier papers such as [16], [17], and [18].

spherically-resolved spaces was obtained in [29, (8.1),8.1]. The v_1-periodic homotopy groups of these spaces were computed in [18], [7], [17], [16], [8], and other papers. We will say no more about these cases.

(2) In Section 2, we use work of Castellana and Broto-Moller to show that the spaces in the infinite families can be decomposed, up to homotopy, as products of factors of p-completions of unitary groups, spheres, and sphere bundles over spheres. See 2.3, 2.5, and 2.7 for the specific results.

(3) In Table 3.2, we list the homotopy types of all cases $(X_n)_p$ which are not products of spheres. There are 31 such cases. In each case, we give the homotopy type as a product of spheres and spaces which are spherically resolved with α_1 attaching maps. In Remark 3.3, we discuss the easily-computed v_1-periodic homotopy groups of these spaces.

(4) The most novel part of the paper is the determination of the v_1-periodic homotopy groups of $(X_{29})_5$, $(X_{31})_5$, and $(X_{34})_7$. We introduce a direct, but nontrivial, path from the invariant polynomials to the v_1-periodic homotopy groups. En route, we determine the Adams operations in $K^*(BX;\hat{\mathbb{Z}}_p)$ and $K^*(X;\hat{\mathbb{Z}}_p)$. In the case of $(X_{34})_7$, we give new explicit formulas for the invariant polynomials. We conjecture in 4.1 (resp. 5.17) that the homotopy type of $(X_{29})_5$ (resp. $(X_{34})_7$) is directly related to $SU(20)$ (resp. $SU(42)$). We explain why it appears that an analogous result is not true for $(X_{31})_5$.

2. Infinite families 2 and 3

Family 3 consists of p-completed[4] spheres S^{2m-1} with $p \equiv 1 \bmod m$, which is a loop space due to work of Sullivan ([34]). The groups $v_1^{-1}\pi_*(S^{2m-1})_{(p)}$, originally due to Mahowald ($p = 2$) and Thompson (p odd), are given in [19, 4.2].

[4] All of our spaces are completed at an appropriate prime p. This will not always be present in our notation. For example, we will often write $SU(n)$ when we really mean its p-completion.

Family 2 consists of spaces $X(m,r,n)$ where $m>1$, $r|m$, and $n>1$. The "degrees" of $X(m,r,n)$ are $m, 2m, \ldots, (n-1)m, \frac{m}{r}n$. These are the degrees of invariant polynomials under a group action used in defining the space. The Clark-Ewing table doubles the degrees to form the "type," as these doubled degrees are the degrees of generators of $H^*(BX;\mathbb{F}_p)$ in the cases which they consider. For most[5] of the irreducible p-compact groups X, $H^*(X;\mathbb{F}_p)$ is an exterior algebra on classes of grading $2d-1$, where d ranges over the degrees. Family 2b consists of spaces $X(m,r,n)$ in which $n=2$ and $r=m$, while family 2a is all other cases. The reason that these are separated is that 2b has more applicable primes. Indeed, for family 2a, there are p-compact groups when $p\equiv 1 \bmod m$, while for family 2b, these exist when $p\equiv \pm1 \bmod m$, and also $p=2$ if $m=4$ or 6, and $p=3$ if $m=3$ or 6. The case $m=6$ in family 2b is the exceptional Lie group G_2. Note that all primes work when $m=6$. The case $(p=2, m=4)$ has $X=Sp(2)$, while $(p=3, m=3)$ has $X=SU(3)$ or $PSU(3)$, the projective unitary group. In this case, there are two inequivalent p-compact groups corresponding to the same Q_p-reflection group; however, since $SU(3)\to PSU(3)$ is a 3-fold covering space, they have isomorphic v_1-periodic homotopy groups.

The following results of Broto and Moller ([11]) and Castellana ([12]) will be useful. They deal with the homotopy fixed-point space X^{hG} when G acts on a space of the same homotopy type as a space X. Here and throughout, C_m denotes a cyclic group of order m, and $U(N)$ is the p-completion of a unitary group.

Theorem 2.1. ([11, 5.2, 5.12]) *If $m|(p-1)$, $0\le s<m$, and $n>0$, then*

$$U(mn+s)^{hC_m}\simeq X(m,1,n)$$

and is a factor in a product decomposition of $U(mn+s)$.

Theorem 2.2. ([11, 5.2, 5.14]) *If $m|(p-1)$, $m\ge 2$, $r>1$, and $n\ge 2$, then*

$$X(m,r,n)^{hC_m}\simeq X(m,1,n-1)$$

and is a factor in a product decomposition of $X(m,r,n)$.

Corollary 2.3. *If $m|(p-1)$ and $r>1$, then*

$$X(m,r,n)\simeq X(m,1,n-1)\times S^{2n\frac{m}{r}-1}$$

[5]According to [31], the only exclusions are certain compact Lie groups when p is very small.

and $X(m,1,n-1)$ is a factor in a product decomposition of $U(m(n-1))$.

Here $X(m,1,1)$ is interpreted as S^{2m-1}.

Proof. We use Theorem 2.2 to get the first factor. By the Kunneth Theorem, the other factor must have the same $\mathbb{F}_p$-cohomology as $S^{2n\frac{m}{r}-1}$, and hence must have the same homotopy type as this sphere. Now we apply Theorem 2.1 to complete the proof. ■

Remark 2.4. Our Corollary 2.3 appears as [12, 1.4], except that she has an apparent typo regarding the dimension of the sphere. Also, neither she nor [11] have the restriction $r > 1$, but it seems that the result is false for $r = 1$, since by induction it would imply that $X(m,1,n)$ is a product of spheres, which is not usually true.

Remark 2.5. Let p be odd. By [29], for any N, p-completed $SU(N)$ splits as a product of $(p-1)$ spaces, each of which has $H^*(-;\mathbb{F}_p)$ an exterior algebra on odd dimensional classes of dimensions b, $b+q, \dots, b+tq$, for some integers b and t. Here and throughout $q = 2(p-1)$. Our space $X(m,1,n-1)$ will be a product of $(p-1)/m$ of these spaces for $SU(m(n-1))$. The v_1-periodic homotopy groups of these spaces can be read off from those of $SU(m(n-1))$, since the $(p-1)$ factors have v_1-periodic homotopy groups in nonoverlapping dimensions. Thus, to the extent that [18] is viewed as being a satisfactory description of $v_1^{-1}\pi_*(SU(n))_{(p)}$,[6] Corollary 2.3 gives $v_1^{-1}\pi_*(X(m,r,n))_{(p)}$ provided $m|(p-1)$.

Example 2.6. *Let $p = 7$. Then $X(2,2,6) \simeq X(2,1,5) \times S^{11}$. There is a product decomposition*

$$(SU(10))_7 \simeq B(3,15) \times B(5,17) \times B(7,19) \times S^9 \times S^{11} \times S^{13},$$

where $B(2n+1, 2n+13)$ denotes a 7-completed S^{2n+1}-bundle over S^{2n+13} with attaching map α_1. Then

$$X(2,1,5) \simeq B(3,15) \times B(7,19) \times S^{11}.$$

[6] [18, 1.4] states that $v_1^{-1}\pi_{2k}(SU(n))_{(p)}$ is a cyclic p-group with exponent $\min(\nu_p(j!S(k,j)) : j \geq n)$, where $S(-,-)$ denotes the Stirling number of the second kind. In [21], more tractable formulas were obtained if $n \leq p^2 - p + 1$. Here and throughout, $\nu_p(-)$ is the exponent of p.

What remains for Family 2 is the cases 2b when $m|(p+1)$. These are the spaces $X(m,m,2)$ with $m|(p+1)$. Let $B(3,2p+1)$ denote the p-completion of an S^3-bundle over S^{2p+1} with attaching map α_1.

Theorem 2.7. *If $m|(p+1)$, then*

$$X(m,m,2)\simeq\begin{cases}B(3,2p+1) & m=p+1\\ S^3\times S^{2m-1} & m<p+1.\end{cases}$$

Proof. Let $X=X(m,m,2)$ with $m|(p+1)$. Then $H^*(X;\mathbb{F}_p)=\Lambda[x_3,x_{2m-1}]$. If $m<p+1$, then by the unstable Adams spectral sequence ([10]), both classes x_3 and x_{2m-1} are spherical. Indeed, the E_2-term begins with towers in dimensions 3 and $2m-1$ emanating from filtration 0, and no possible differentials. See Diagram 3.4. Because X is an H-space, the maps $S^3\to X$ and $S^{2m-1}\to X$ yield a map $S^3\times S^{2m-1}\to X$, and it is a p-equivalence by Whitehead's Theorem.

On the other hand, suppose $m=p+1$. We will show that $\mathcal{P}^1(x_3)=x_{2p+1}$. It then follows from [29, 7.1] that there is a p-equivalence $B(3,2p+1)\to X$.

To see that $\mathcal{P}^1(x_3)=x_{2p+1}$, we use the classifying space BX, which satisfies $H^*(BX;\mathbb{F}_p)=\mathbb{F}_p[y_4,y_{2p+2}]$. We will prove that $\mathcal{P}^1(y_4)=y_{2p+2}+Ay_4^{(p+1)/2}$, for some generator y_{2p+2} and some $A\in\mathbb{F}_p$, from which the desired result about the x's follows immediately from the map $\Sigma X\to BX$, which in $H^*(-;\mathbb{F}_p)$ sends y_{j+1} to x_j and sends products to 0.

First note that

$$\begin{aligned}\mathcal{P}^1(y_4)&=Ay_4^{(p+1)/2}+By_{2p+2}\\ \mathcal{P}^1(y_{2p+2})&=Cy_4^p+Dy_4^{(p-1)/2}y_{2p+2},\end{aligned}$$

for some A, B, C, D in $\mathbb{F}_p$. By the unstable property of the Steenrod algebra,

$$\mathcal{P}^{p+1}(y_{2p+2})=y_{2p+2}^p.\tag{2.8}$$

We must have

$$\mathcal{P}^p(y_{2p+2})=\sum_{j=0}^{p-1}c_jy_4^{1+j(p+1)/2}y_{2p+2}^{p-1-j},$$

for some $c_j\in\mathbb{F}_p$. Since $\mathcal{P}^{p+1}=\mathcal{P}^1\mathcal{P}^p$ and

$$\mathcal{P}^1(y_4^iy_{2p+2}^j)=i\mathcal{P}^1(y_4)y_4^{i-1}y_{2p+2}^j+jy_4^i\mathcal{P}^1(y_{2p+2})y_{2p+2}^{j-1},\tag{2.9}$$

the only way to obtain (2.8) is if $c_0B = 1$ in $\mathbb{F}_p$. Thus B must be a unit, and the generator y_{2p+2} can be chosen so that $B = 1$. ■

3. Nonmodular individual cases

In this section, we consider all cases 4 through 34, excluding case 28 (which is F_4), in the Shephard-Todd numbering in which p does not divide the order of the reflection group. We obtain a very attractive result. One modular case, $(X_{12}, p = 3)$ is also included here. There is some overlap of our methods and results here with those in [27].

Theorem 3.1. *Let $X = (X_n)_p$ with $4 \le n \le 34$ and $n \ne 28$, excluding the modular cases $(X_{29})_5$, $(X_{31})_5$, and $(X_{34})_7$, which will be considered in the next two sections. Then $X \simeq \prod S^{2d-1}$, where $2d$ ranges over the integers listed as the "type" in [13], except for the 31 cases listed in Table 3.2. In these, each $B(-, \ldots, -)$ is built by fibrations from spheres of the indicated dimensions, with α_1 as each attaching map, and occurs as a factor in a product decomposition of the p-completion of some $SU(N)$.*

We will call the integers d, which are $1/2$ times the "type " numbers of Clark-Ewing, the "degrees."

Table 3.2. Cases in Theorem 3.1 which are not products of spheres

Case	*Prime*	*Space*
5	7	$B(11, 23)$
8	5	$B(15, 23)$
9	17	$B(15, 47)$
10	13	$B(23, 47)$
12	3	$B(11, 15)$
14	19	$B(11, 47)$
16	11	$B(39, 59)$
17	41	$B(39, 119)$
18	31	$B(59, 119)$
20	19	$B(23, 59)$
24	11	$B(7, 27) \times S^{11}$
25	7	$B(11, 23) \times S^{17}$
26	7	$B(11, 23, 35)$
26	13	$B(11, 35) \times S^{23}$
27	19	$B(23, 59) \times S^{13}$
29	13	$B(15, 39) \times S^7 \times S^{23}$
29	17	$B(7, 39) \times S^{15} \times S^{23}$
30	11	$B(3, 23) \times B(39, 59)$
30	19	$B(3, 39) \times S^{27} \times S^{59}$
30	29	$B(3, 59) \times S^{23} \times S^{39}$
31	13	$B(15, 39) \times B(23, 47)$
31	17	$B(15, 47) \times S^{23} \times S^{39}$
32	7	$B(23, 35, 47, 59)$
32	13	$B(23, 47) \times B(35, 59)$
32	19	$B(23, 59) \times S^{35} \times S^{47}$
33	7	$B(7, 19) \times B(11, 23, 35)$
33	13	$B(11, 35) \times S^7 \times S^{19} \times S^{23}$
34	13	$B(11, 35, 59, 83) \times B(23, 47)$
34	19	$B(11, 47, 83) \times B(23, 59) \times S^{35}$
34	31	$B(23, 83) \times S^{11} \times S^{35} \times S^{47} \times S^{59}$
34	37	$B(11, 83) \times S^{23} \times S^{35} \times S^{47} \times S^{59}$

Remark 3.3. The v_1-periodic homotopy groups of $B(2n+1, 2n+2p-1)$ were obtained in [8, 1.3]. Those of $B(11, 23, 35)_7$ and $B(23, 35, 47, 59)_7$ were obtained in [8, 1.4]. Using [21, 1.5,1.9], we find that for $\epsilon = 0, 1$,

$$v_1^{-1}\pi_{2t-\epsilon}(B(11, 35, 59, 83))_{(13)} \approx \begin{cases} 0 & t \not\equiv 5 \quad (12) \\ \mathbb{Z}/13^{\max(f_5(t), f_{17}(t), f_{29}(t), f_{41}(t))} & t \equiv 5 \quad (12), \end{cases}$$

where $f_\gamma(t)=\min(\gamma,4+\nu_{13}(t-\gamma))$, while

$$v_1^{-1}\pi_{2t-\epsilon}(B(11,47,83))_{(19)}\approx\begin{cases}0 & t\not\equiv 5 \quad (18)\\ \mathbb{Z}/19^{\max(f'_5(t),f'_{23}(t),f'_{41}(t))} & t\equiv 5 \quad (18),\end{cases}$$

where $f'_\gamma(t)=\min(\gamma,3+\nu_{19}(t-\gamma))$.

Proof of Theorem 3.1. It is straightforward to check that the pairs (case, prime) listed in Table 3.2 are the only non-modular cases in [2, Table 1] in which an admissible prime p satisfies that $(p-1)$ divides the difference of distinct degrees. Indeed all other admissible primes have $(p-1)$ greater than the maximum difference of degrees. For example, Case 30 requires $p\equiv 1,4 \bmod 5$, and the degrees are 2, 12, 20, 30. The first few primes of the required congruence are 11, 19, and 29. Clearly 10, 18, and 28 divide differences of these degrees, but no larger $(p-1)$ can. Thus the unstable Adams spectral sequence argument used in proving Theorem 2.7 works the same way here to show that X is a product of S^{2d-1} in all cases not appearing in Table 3.2. In the relevant range, the E_2-term will consist only of infinite towers, one for each generator. The first deviation from that is a $\mathbb{Z}/p$ in filtration 1 in homotopy dimension $(2d-1)+(2p-3)$, where d is the smallest degree. This will always be greater than the dimension of the largest S^{2d-1}.

The next step is to show that the Steenrod operation $\mathcal{P}^1$ in $H^*(X;\mathbb{F}_p)$ must connect all the classes listed as adjacent generators in one of the B-spaces in Table 3.2. We accomplish this by considering the A-module $H^*(BX;\mathbb{F}_p)$. All cases involving factors of $B(2m-1,2m+2p-3)$ are implied by Lemma 3.7 by applying $H^*(BX)\to H^{*-1}(X)$, which sends products to 0. Similarly, Lemma 3.9 covers the cases with a factor $B(11,23,35)$ or $B(11,47,83)$. Finally, Lemma 3.11 covers the cases with a factor $B(23,35,47,59)$ or $B(11,35,59,83)$.

Now we must show that the spaces X have the homotopy type claimed. The first 10 cases are immediate from [29, 7.1], and the two other non-product cases, i.e. $(X_{26})_7$ and $(X_{32})_7$, follow from [29, 7.2,7.6]. Note that these results of [29] did not deal with p-completed spaces, but the obstruction theory arguments used there apply in the p-complete context. There are 15 additional types which we claim to be quasi p-regular. As defined in [30], a space is quasi p-regular if it is p-equivalent to a product of spheres and spaces of the form $B(2n+1,2n+2p-1)$. In [30] (see esp.

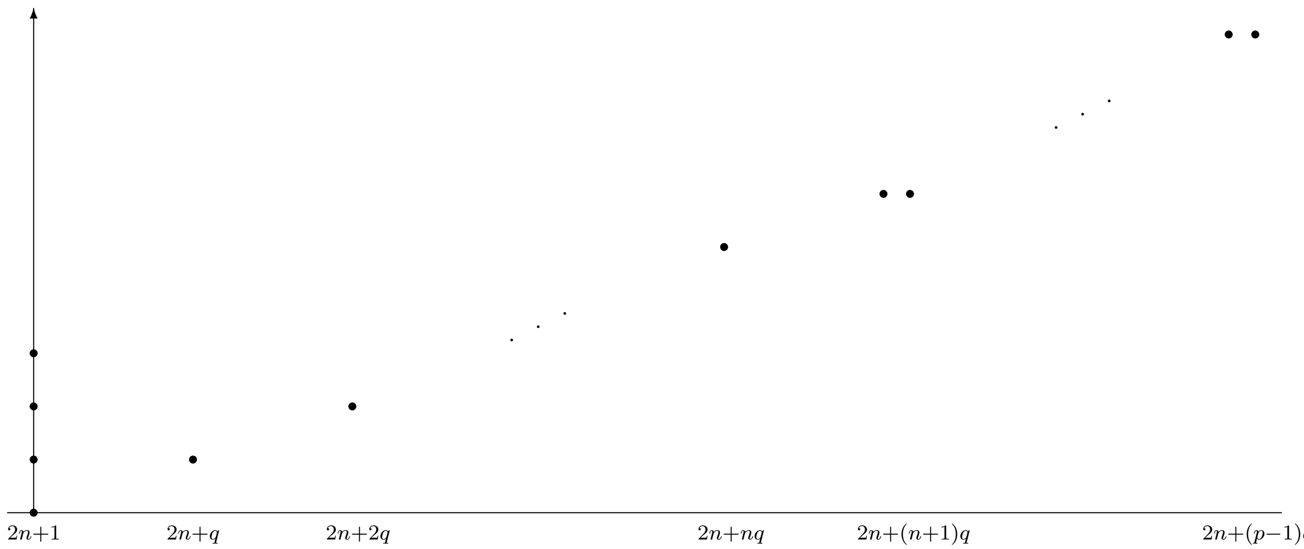
[30, pp. 330-334]), many exceptional Lie groups are shown to be quasi p-regular (for appropriate p) using a skeletal approach. We could use that approach here, but we prefer to use the unstable Adams spectral sequence (UASS). The two methods are really equivalent.

Let $q = 2p-2$. In Diagrams 3.4 and 3.5, we illustrate the UASS for S^{2n+1} in dimension less than $2n+pq-1$ and for $B(2n+1, 2n+q+1)$ in dimension less than $2n+3q-3$. Diagram 3.4 gives a nice interpretation of the statement of the homotopy groups in [35, 13.4]. If $n \geq p$, the paired dots in Diagram 3.4 will not occur in the pictured range. The nice thing about these charts is that the $\mathbb{F}_p$-cohomology groups of our spaces X are known to agree with that of their putative product decomposition as unstable algebras over the Steenrod algebra, and are of the required universal form for the UASS to apply; hence their UASS has E_2-term the sum of the relevant charts of spheres and B-spaces. In all cases, there will be no possible differentials.

One can check that in all 15 cases in which X is claimed to be quasi p-regular, the towers in UASS(X) corresponding to the spheres and the bottom cell of each $B(2n+1, 2n+q+1)$ cannot support a differential, and hence yield maps from the sphere or S^{2n+1} into X. Next one checks that $\pi_{2n+q}(X) = 0$ and $\pi_{4n+q+1}(X) = 0$. As these are the groups in which the obstruction to extending the map $S^{2n+1} \to X$ over $B(2n+1, 2n+q+1)$ lie, we obtain the desired extension. Finally, we take the product of maps $B \to X$ and $S^{2d_i-1} \to X$, using the group structure of X, to obtain the desired p-equivalence from a product of spheres and B-spaces into X.

The remaining cases, $(X_{33})_7$, $(X_{34})_{13}$, and $(X_{34})_{19}$, are handled similarly. The E_2-term of the UASS converging to $\pi_*(X)$ is isomorphic to that of its putative product decomposition. For example, $E_2(X_{34})_{13}$ is the sum of Diagram 3.5 with $n = 11$ and $q = 24$ plus Diagram 3.6. We can map $S^{23} \to X$ and $S^{11} \to X$ corresponding to generators of homotopy groups. Then we can extend the first map over the 47- and 70-cells because $\pi_{46}(X) = 0$ and $\pi_{69}(X) = 0$. This gives a map $B(23, 47) \to X$. Similarly we can extend the second map over cells of $B(11, 35, 59, 83)$ of dimension 46, 70, 94, 118, 142, 105, 129, 153, and 188. Taking the product of these two maps, using the multiplication of X, yields the desired 13-equivalence $B(23, 47) \times B(11, 35, 59, 83) \to X$. The other two cases are handled similarly. ■

Diagram 3.4. *$UASS(S^{2n+1})$ in dim $< 2n + pq - 1$. Here $n < p$.*



Where there is a pair of dots, the grading at the bottom refers to the one on right, and the other is in grading 1 less.

Diagram 3.5. *$UASS(B(2n + 1, 2n + q + 1))$ in dim $< 2n + 3q - 3$.*

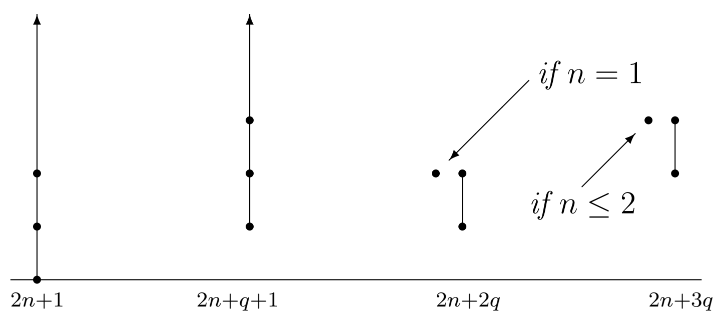
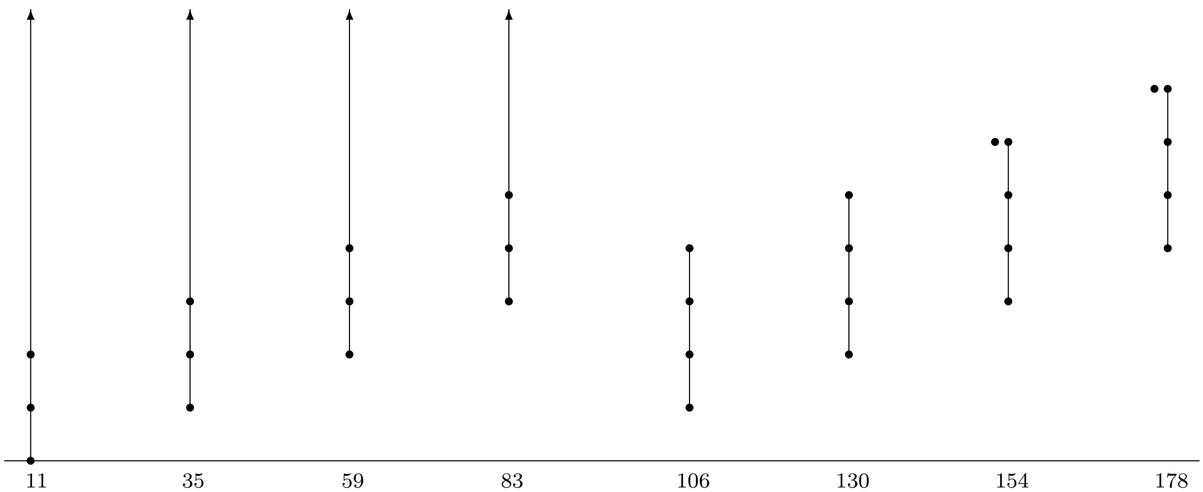


Diagram 3.6. *$UASS(B(11,35,59,83)_{13})$ in dim <200.*



In the following lemmas, which were used above, g_i denotes a generator in grading i.

Lemma 3.7. *a. If $m \not\equiv 1$ mod p and $\mathbb{F}_p[g_{2m}, g_{2m+2p-2}]$ is an unstable A-algebra, then $\mathcal{P}^1 g_{2m} \equiv u g_{2m+2p-2}$ mod decomposables, with $u \neq 0$.*

b. The same conclusion holds if the unstable A-algebra contains additional generators in dimensions $\not\equiv 2m$ mod $(2p-2)$.

Proof. a. For dimensional reasons, we must have $\mathcal{P}^1 g_{2m} = \alpha g_{2m+2p-2}$ plus possibly a power of g_{2m}, for some $\alpha \in \mathbb{F}_p$, and $\mathcal{P}^1 g_{2m+2p-2} = g_{2m} Y$, for some polynomial Y. The unstable condition requires that $\mathcal{P}^{m+p-1} g_{2m+2p-2} = g^p_{2m+2p-2}$, and, since $m \not\equiv 1$ mod p, this equals, up to unit, $\mathcal{P}^1 \mathcal{P}^{m+p-2} g_{2m+2p-2}$. For dimensional reasons,

$$\mathcal{P}^{m+p-2} g_{2m+2p-2} = \beta g_{2m} g^{p-1}_{2m+2p-2} + g^3_{2m} Z \tag{3.8}$$

for some $\beta \in \mathbb{F}_p$ and some polynomial Z. By the Cartan formula (similar to (2.9)), the only way that $\mathcal{P}^1$ applied to (3.8) can yield $g^p_{2m+2p-2}$ is if both α and β are units.

b. The only way that the additional generators could affect any of the considerations of part (a) would be if several of them (possibly the same one) were multiplied together to get into the congruence of part (a). By the Cartan formula, $\mathcal{P}^1$ of such a product will still involve some of these additional generators as factors, and so cannot yield the $g^p_{2m+2p-2}$ term on which the argument focuses. ∎

Lemma 3.9. *a. If* $\mathbb{F}_{19}[g_{12}, g_{48}, g_{84}]$ *is an unstable* A*-algebra, then, mod decomposables,* $\mathcal{P}^1 g_{12} \equiv u_1 g_{48}$ *and* $\mathcal{P}^1 g_{48} \equiv u_2 g_{84}$ *with* $u_i \neq 0$.

b. If $\mathbb{F}_7[g_{12}, g_{24}, g_{36}]$ *is an unstable* A*-algebra, then, mod decomposables,* $\mathcal{P}^1 g_{12} = u_1 g_{24}$ *and* $\mathcal{P}^1 g_{24} = u_2 g_{36}$ *with* $u_i \neq 0$.

c. The same conclusion holds if the unstable A*-algebra contains additional generators in dimensions* $\not\equiv 12$ *mod* $(2p-2)$.

Proof. a. For dimensional reasons, we must have

$$\begin{aligned}
\mathcal{P}^1 g_{12} &= \alpha_1 g_{48} + \alpha_2 g_{12}^4 \\
\mathcal{P}^1 g_{48} &= \beta_1 g_{84} + \beta_2 g_{12}^3 g_{48} + \beta_3 g_{12}^7 \\
\mathcal{P}^1 g_{84} &= \gamma_1 g_{12}^3 g_{84} + \gamma_2 g_{12}^2 g_{48}^2 + \gamma_3 g_{12}^6 g_{48} + \gamma_4 g_{12}^{10}
\end{aligned}$$

for some coefficients α_i, β_i, γ_i. The unstable condition requires, up to unit,

$$\mathcal{P}^1 \mathcal{P}^{41} g_{84} = g_{84}^{19} \text{ and } \mathcal{P}^1 \mathcal{P}^{23} g_{48} = g_{48}^{19}. \tag{3.10}$$

We use the Cartan formula as in the previous proof. The only terms in $\mathcal{P}^1 g_i$ involving just g_{84} or just g_{48} are $\beta_1 g_{84}$ and $\alpha_1 g_{48}$, and so these must be nonzero in order to obtain (3.10).

b. We work mod the ideal generated by g_{12}. Then $\mathcal{P}^1 g_{12} \equiv \alpha g_{24}$, $\mathcal{P}^1 g_{24} \equiv \beta g_{36}$, and $\mathcal{P}^1 g_{36} \equiv \gamma g_{24}^2$, for some α, β, γ in $\mathbb{F}_7$. This latter term complicates things somewhat.

That $\mathcal{P}^1 \mathcal{P}^{17} g_{36} = u g_{36}^7$ implies $\beta \neq 0$ as before. But there are two ways that $\mathcal{P}^1 \mathcal{P}^{11} g_{24}$ might yield g_{24}^7, one via $\mathcal{P}^1(g_{12} g_{24}^6)$ and the other via $\mathcal{P}^1(g_{24}^5 g_{36})$. Instead, we consider $(\mathcal{P}^1)^5 \mathcal{P}^7 g_{24}$. We must have

$$\mathcal{P}^7 g_{24} \equiv \delta_1 g_{36}^3 + \delta_2 g_{24}^3 g_{36}$$

for some $\delta_i \in \mathbb{F}_7$. We compute

$$\begin{aligned}
(\mathcal{P}^1)^5 g_{36}^3 &\equiv \beta \gamma^4 g_{24}^7 - \beta^3 \gamma^2 g_{24} g_{36}^4 \\
(\mathcal{P}^1)^5 (g_{24}^3 g_{36}) &\equiv \beta^2 \gamma^3 g_{24}^7 + 5\beta^3 \gamma^2 g_{24}^4 g_{36}^2.
\end{aligned}$$

Assuming that $\alpha = 0$, so that the omitted terms of the form $g_{12}Y$ in $\mathcal{P}^7 g_{24}$ have $(\mathcal{P}^1)^5(\text{them}) \equiv 0$, then we obtain

$$\begin{aligned} ug_{24}^7 &\equiv (\mathcal{P}^1)^5\mathcal{P}^7(g_{24}) \\ &\equiv (\mathcal{P}^1)^5(\delta_1 g_{36}^3 + \delta_2 g_{24}^3 g_{36}) \\ &\equiv \delta_1(\beta\gamma^4 g_{24}^7 - \beta^3\gamma^2 g_{24}g_{36}^4) + \delta_2(\beta^2\gamma^3 g_{24}^7 + 5\beta^3\gamma^2 g_{24}^4 g_{36}^2). \end{aligned}$$

Coefficients of g_{24}^7 imply $\beta \neq 0$, $\gamma \neq 0$, and some $\delta_i \neq 0$, but this then gives a contradiction regarding $g_{24}g_{36}^4$ or $g_{24}^4 g_{36}^2$. Thus the assumption that $\alpha = 0$ must have been false.

c. This follows by the argument used for part (b) in Lemma 3.7. ■

Lemma 3.11. *a. If* $\mathbb{F}_{13}[g_{12}, g_{36}, g_{60}, g_{84}]$ *is an unstable A-algebra, then, mod decomposables,* $\mathcal{P}^1 g_{12} \equiv u_1 g_{36}$, $\mathcal{P}^1 g_{36} \equiv u_2 g_{60}$, *and* $\mathcal{P}^1 g_{60} \equiv u_3 g_{84}$ *with* $u_i \neq 0$.

b. If $\mathbb{F}_7[g_{24}, g_{36}, g_{48}, g_{60}]$ *is an unstable A-algebra, then, mod decomposables,* $\mathcal{P}^1 g_{24} = u_1 g_{36}$, $\mathcal{P}^1 g_{36} = g_{48}$, *and* $\mathcal{P}^1 g_{48} = g_{60}$ *with* $u_i \neq 0$.

Proof. a. We can easily prove the second and third parts of the conclusion as in the proofs of the preceding lemmas. That $\mathcal{P}^1 g_{84}$ might involve g_{36}^3 complicates the proof of the first part.

Assume that

$$\mathcal{P}^1 g_{12} = 0 g_{36} + \alpha g_{12}^3. \tag{3.12}$$

Under this hypothesis, we may work mod the ideal generated by g_{12}. After possibly varying generators by a unit, we have

$$\mathcal{P}^1 g_{36} \equiv g_{60}, \qquad \mathcal{P}^1 g_{60} \equiv g_{84}, \qquad \mathcal{P}^1 g_{84} \equiv \gamma g_{36}^3,$$

with $\gamma \in \mathbb{F}_p$.

We must have $(\mathcal{P}^1)^5\mathcal{P}^{13} g_{36} = g_{36}^{13}$. There are six monomials (not involving g_{12}) which might be part of $\mathcal{P}^{13} g_{36}$. They label the columns of Table 3.13, whose columns give the results of a `Maple` computation of $(\mathcal{P}^1)^5$ (mod g_{12}) applied to each. We must have that g_{36}^{13} can be obtained as a linear combination of the columns of Table 3.13 considered as a matrix M. Clearly this cannot be done if $\gamma = 0$. So we may assume that γ is a unit.

Equivalently, we must have that $e_1 := (1,0,0,0,0,0,0,0,0,0)$ is in the row space of M^T. Divide the rows of M^T by γ^2, γ^3, γ^3, γ^4, γ^3, and γ^4, resp. This does not change the row space. Now multiply the columns of the obtained matrix by 1, γ, γ, γ^2, γ^2, γ^2, γ^3, γ^3, and γ^3, resp. This does not change whether e_1 is in the row space, and yields the numerical matrix obtained by transposing Table 3.13 and ignoring the γ's. `Maple` easily rowreduces this matrix mod 13, obtaining

$$\begin{pmatrix} 1&0&0&0&0&0&3&1&11&0\\ 0&1&0&0&0&0&7&9&3&0\\ 0&0&1&0&0&0&3&11&8&10\\ 0&0&0&1&0&0&0&4&0&0\\ 0&0&0&0&1&0&7&5&10&0\\ 0&0&0&0&0&1&6&0&11&9 \end{pmatrix}$$

Since e_1 is not in the row space of this matrix, we conclude that $(\mathcal{P}^1)^5\mathcal{P}^{13}g_{36}$ cannot equal g_{36}^{13} under the hypothesis (3.12), and hence $\mathcal{P}^1 g_{12} \equiv u g_{36}$ mod decomposables, with u a unit, completing the proof of (a).

The proof of part (b) is very similar and is omitted. ■

Table 3.13. *Action of* $(\mathcal{P}^1)^5$

	$g_{36}^8 g_{60}$	$g_{36}^3 g_{60}^4$	$g_{36}^4 g_{60}^2 g_{84}$	$g_{60}^3 g_{84}^2$	$g_{36}^5 g_{84}^2$	$g_{36} g_{60} g_{84}^3$
g_{36}^{13}	$5\gamma^2$	0	$4\gamma^3$	0	0	$8\gamma^4$
$g_{36}^9 g_{60} g_{84}$	8γ	$9\gamma^2$	$6\gamma^2$	$7\gamma^3$	$2\gamma^2$	0
$g_{36}^8 g_{60}^3$	5γ	$5\gamma^2$	$11\gamma^2$	$9\gamma^3$	$9\gamma^2$	$7\gamma^3$
$g_{36}^6 g_{84}^3$	8	6γ	0	$11\gamma^2$	7γ	$12\gamma^2$
$g_{36}^5 g_{60}^2 g_{84}^2$	1	2γ	6γ	$6\gamma^2$	8γ	$9\gamma^2$
$g_{36}^4 g_{60}^4 g_{84}$	6	7γ	5γ	$7\gamma^2$	12γ	$8\gamma^2$
$g_{36}^3 g_{60}^6$	12	4γ	3γ	$4\gamma^2$	0	0
$g_{36}^2 g_{60} g_{84}^4$	0	1	1	2γ	3	9γ
$g_{36} g_{60}^3 g_{84}^3$	0	8	5	9γ	4	10γ
$g_{60}^5 g_{84}^2$	0	9	12	10γ	3	12γ

4. 5-primary modular cases

In this section, we determine the v_1-periodic homotopy groups of the modular 5-compact groups X_{29} and X_{31}. We pass directly from invariant polynomials to Adams operations in $K^*(X)$ and thence to $v_1^{-1}\pi_*(X)$. We provide strong evidence for a conjecture (4.1) about the homotopy type of $(X_{29})_5$; however, as we will explain, an analogue for $(X_{31})_5$ seems unlikely.

One of the factors in the product decomposition of $SU(20)_5$ given in [29] is an H-space $B^5_3(5)$ whose $\mathbb{F}_5$-cohomology is an exterior algebra on classes of grading 7, 15, 23, 31, and 39, and which is built from spheres of these dimensions by fibrations. By [36], there is a product decomposition

$$(SU(20)/SU(15))_5 \simeq S^{31} \times S^{33} \times S^{35} \times S^{37} \times S^{39}.$$

Let $B(7, 15, 23, 39)$ denote the fiber of the composite

$$B^5_3(5) \to SU(20)_5 \to (SU(20)/SU(15))_5 \xrightarrow{\rho} (S^{31})_5.$$

Conjecture 4.1. *There is an equivalence*

$$(X_{29})_5 \simeq B(7, 15, 23, 39).$$

As we shall prove in 4.16, the evidence for this conjecture is that the two spaces have isomorphic Adams modules $K^*(X; \hat{\mathbb{Z}}_5)$ and hence isomorphic v_1-periodic homotopy groups. The obstruction-theoretic method which we used to construct most of the equivalences of 3.2 does not work in this case. There are several possible obstructions to extending maps over cells in this case.

The input to determining the Adams module $K^*(X_{29}; \hat{\mathbb{Z}}_5)$ is the following result due to Aguadé ([1]) and Maschke ([28]). Throughout the rest of the paper, we will denote by $m_{(e_1,\dots,e_k)}$ the smallest symmetric polynomial on variables $x_1, \dots, x_\ell$ (the value of ℓ will be implicit) containing the term $x_1^{e_1} \cdots x_k^{e_k}$.

Theorem 4.2. *There is a reflection group G_{29} acting on $(\hat{\mathbb{Z}}_5)^4$, and there is a space BX_{29} and map $BT \to BX_{29}$ with $BT = K((\hat{\mathbb{Z}}_5)^4, 2)$ such that*

$$H^*(BX_{29}; \hat{\mathbb{Z}}_5) \approx H^*(BT; \hat{\mathbb{Z}}_5)^{G_{29}},$$

the invariants under the natural action of G_{29} on $H^*(BT; \hat{\mathbb{Z}}_5) = \hat{\mathbb{Z}}_5[x_1, x_2, x_3, x_4]$ with $|x_i| = 2$. Moreover, $H^*(BT; \hat{\mathbb{Z}}_5)^{G_{29}}$ is a polynomial algebra on the following four *invariant polynomials:*

$$\begin{aligned}
f_4 &= m_{(4)} - 12m_{(1,1,1,1)} \\
f_8 &= m_{(8)} + 14m_{(4,4)} + 168m_{(2,2,2,2)} \\
f_{12} &= m_{(12)} - 33m_{(8,4)} + 330m_{(4,4,4)} + 792m_{(6,2,2,2)} \\
f_{20} &= m_{(20)} - 19m_{(16,4)} - 494m_{(12,8)} - 336m_{(14,2,2,2)} + 716m_{(12,4,4)} \\
&\quad +1038m_{(8,8,4)} + 7632m_{(10,6,2,2)} + 129012m_{(8,4,4,4)} + 106848m_{(6,6,6,2)}.
\end{aligned}$$

Proof. The group G_{29} is the subgroup of $GL(\mathbf{C},4)$ generated by the following four matrices. These can be seen explicitly in [1].

$$\frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $i \in \hat{\mathbb{Z}}_5$, these act on $(\hat{\mathbb{Z}}_5)^4$, and this induces an action on

$$H^*(K((\hat{\mathbb{Z}}_5)^4, 2)) \approx \hat{\mathbb{Z}}_5[x_1, x_2, x_3, x_4].$$

The invariants of this action were determined by Maschke ([28]) to be the polynomials stated in the theorem. Although he did not state them all explicitly, they can be easily generated by: (a) define ϕ, ψ_i, and χ as on his page 501, then (b) define $\Phi_1, \dots, \Phi_6$ as on his page 504, and finally (c) let $f_4 = -\frac{1}{2}\Phi_6$ and $f_8 = F_8$, $f_{12} = F_{12}$, and $f_{20} = F_{20}$ as on his page 505. See also [33, p.287] for a reference to this work.

Actually, Maschke's work and that of [33] involved finding generators for the complex invariant ring. To see that these integral polynomials generate the invariant ring over $\hat{\mathbb{Z}}_5$, one must show that they cannot be decomposed over $\mathbb{Z}/5$. For example, one must verify that f_{20} cannot be decomposed mod 5 as a linear combination of f_8f_{12}, $f_4^2f_{12}$, $f_4f_8^2$, $f_4^3f_8$, and f_4^5. The need to do this was pointed out to the author by Kasper Andersen in a dramatic way, as will be described prior to 5.6. The verification here was performed by Andersen using a `Magma` program.

Aguadé ([1]) constructed the 5-compact group (BX, X) corresponding to this modular reflection group. ■

The approach based on the following proposition benefits from a suggestion of Clarence Wilkerson.

Proposition 4.3. *Let (BX, X) be a p-compact group corresponding to a reflection group G acting on $BT = K((\hat{\mathbb{Z}}_p)^n, 2)$. Suppose $H^*(BT; \hat{\mathbb{Q}}_p)^G = \hat{\mathbb{Q}}_p[f_1, \dots, f_k]$, where f_i is a polynomial in $y_1, \dots, y_n$ with $y_j \in H^2(BT; \hat{\mathbb{Q}}_p)$ corresponding to the j^{th} factor. Let $K^*(BT; \hat{\mathbb{Q}}_p) = \hat{\mathbb{Q}}_p[\![x_1, \dots, x_n]\!]$ with x_i the class of $H - 1$ in the i^{th} factor, where H is the complex Hopf bundle. Let $\ell_0(x) = \log(1 + x)$. Then*

$$K^*(BX; \hat{\mathbb{Z}}_p) \approx \hat{\mathbb{Q}}_p[\![f_1(\ell_0(x_1), \dots, \ell_0(x_n)), \dots, f_k(\ell_0(x_1), \dots, \ell_0(x_n))]\!] \cap \hat{\mathbb{Z}}_p[\![x_1, \dots, x_n]\!].$$

Proof. The Chern character $K^*(BT; \hat{\mathbb{Q}}_p) \xrightarrow{\mathrm{ch}} H^*(BT; \hat{\mathbb{Q}}_p)$ satisfies $\mathrm{ch}(\ell_0(x_i)) = y_i$ and hence, since ch is a ring homomorphism, $\mathrm{ch}(f_j(\ell_0(x_1), \dots, \ell_0(x_n))) = f_j(y_1, \dots, y_n)$. It commutes with the action of G, and hence sends invariants to invariants. Indeed

$$K^*(BT; \hat{\mathbb{Q}}_p)^G = \hat{\mathbb{Q}}_p[\![f_1(\ell_0(x_1), \dots, \ell_0(x_n)), \dots, f_k(\ell_0(x_1), \dots, \ell_0(x_n))]\!]. \tag{4.4}$$

The invariant ring in $K^*(BT; \hat{\mathbb{Z}}_p)$ is just the intersection of (4.4) with $\hat{\mathbb{Z}}_p[\![x_1, \dots, x_n]\!]$. Finally we use a result of [24] that $K^*(BX; \hat{\mathbb{Z}}_p) \approx K^*(BT; \hat{\mathbb{Z}}_p)^G$. ■

Thus with f_4, f_8, f_{12}, f_{20} as in 4.2, we wish to find algebraic combinations of

$$f_4(\ell_0(x_1), \dots, \ell_0(x_4)), \dots, f_{20}(\ell_0(x_1), \dots, \ell_0(x_4))$$

which have coefficients in $\hat{\mathbb{Z}}_5$. A theorem of [24] which states that for a p-compact group BX there is an isomorphism $K^*(BX; \hat{\mathbb{Z}}_p) \approx \hat{\mathbb{Z}}_p[\![g_1, \dots, g_k]\!]$, and the collapsing, for dimensional reasons, of the Atiyah-Hirzebruch spectral sequence

$$H^*(BX; K^*(\mathrm{pt}; \hat{\mathbb{Z}}_p)) \Rightarrow K^*(BX; \hat{\mathbb{Z}}_p) \tag{4.5}$$

implies that the generators g_j can be chosen to be of the form $f_j(x_1, \dots, x_n)$ mod higher degree polynomials.

Finding these algebraic combinations can be facilitated by using the p-typical log series

$$\ell_p(x) = \sum_{i \geq 0} x^{p^n}/p^n.$$

By [23], there is a series $h(x) \in \mathbb{Z}_{(p)}[\![x]\!]$ such that $\ell_0(h(x)) = \ell_p(x)$ and $h(x) \equiv x \bmod (x^2)$. Let $x_i' = h(x_i)$. For any $c_{\mathbf{e}} \in \hat{\mathbb{Q}}_p$ with $\mathbf{e} = (e_4, e_8, e_{12}, e_{20})$, we have

$$\begin{aligned} & \sum c_{\mathbf{e}} f_4(\ell_p(x_1), \ldots, \ell_p(x_4))^{e_4} \cdots f_{20}(\ell_p(x_1), \ldots, \ell_p(x_4))^{e_{20}} && (4.6)\\ = & \sum c_{\mathbf{e}} f_4(\ell_0(x_1'), \ldots, \ell_0(x_4'))^{e_4} \cdots f_{20}(\ell_0(x_1'), \ldots, \ell_0(x_4'))^{e_{20}}, && (4.7) \end{aligned}$$

where the sums are taken over various $\mathbf{e}$. We will find $c_{\mathbf{e}}$ so that (4.6) is in $\hat{\mathbb{Z}}_p[\![x_1, \ldots, x_4]\!]$. Thus so is (4.7), and hence also $\sum c_{\mathbf{e}} f_4(\ell_0(x_1), \ldots, \ell_0(x_4))^{e_4} \cdots f_{20}(\ell_0(x_1), \ldots, \ell_0(x_4))^{e_{20}}$, since $h(x) \in \mathbb{Z}_{(p)}[\![x]\!]$.

A `Maple` program, which will be described in the proof, was used to prove the following result.

Theorem 4.8. *Let f_4, f_8, f_{12}, f_{20} be as in 4.2, and let*

$$F_j = F_j(x_1, \ldots, x_4) = f_j(\ell_0(x_1), \ldots, \ell_0(x_4)).$$

Then the following series are 5-integral through grading 20; i.e., their coefficients of all monomials $x_1^{e_1} \cdots x_4^{e_4}$ with $\sum e_i \le 20$ are 5-integral.

$$\begin{aligned} & F_4 - \tfrac{1}{10} F_4^2 - \tfrac{1}{5} F_8 - \tfrac{16}{25} F_{12} - \tfrac{7}{25} F_4 F_8 + \tfrac{4}{25} F_4^3 - \tfrac{13}{125} F_4 F_{12} - \tfrac{57}{125} F_4^2 F_8 \\ & \quad - \tfrac{102}{125} F_4^4 - \tfrac{62}{125} F_8^2 - \tfrac{64}{125} F_{20} - \tfrac{4}{625} F_4^5 - \tfrac{42}{125} F_4^2 F_{12} - \tfrac{11}{25} F_4 F_8^2 - \tfrac{72}{125} F_8 F_{12}; \\ & F_8 - \tfrac{8}{5} F_{12} - \tfrac{7}{25} F_8^2 - \tfrac{4}{25} F_{20} - \tfrac{21}{125} F_8 F_{12}; \\ & F_{12} - \tfrac{2}{5} F_8^2 - \tfrac{1}{5} F_{20} - \tfrac{4}{25} F_8 F_{12}. \end{aligned}$$

Proof. As observed in the paragraph preceding the theorem, it suffices to show that the same is true for $\widetilde{F}_j = f_j(x_1 + \frac{1}{5}x_1^5, \ldots, x_4 + \frac{1}{5}x_4^5)$. The advantage of this is to decrease the number of terms which must be kept track of and looked at. We work one grading at a time, expanding relevant products of F_j's as combinations of monomial symmetric polynomials in the fixed grading, and then solving a system of linear equations to find the combinations that work. We illustrate with the calculation for modifications of F_4 in gradings 8 and then 12.

In grading 8, we have

$$\begin{aligned} \widetilde{F}_4 &= \tfrac{4}{5} m_{(8)} - \tfrac{12}{5} m_{(5,1,1,1)} \\ \widetilde{F}_8 &= m_{(8)} + 14 m_{(4,4)} + 168 m_{(2,2,2,2)} \\ \widetilde{F}_4^2 &= m_{(8)} - 24 m_{(5,1,1,1)} + 2 m_{(4,4)} + 144 m_{(2,2,2,2)}. \end{aligned}$$

We wish to choose a and b so that, in grading 8, $\frac{a}{5}\widetilde{F}_8 + \frac{b}{5}\widetilde{F}_4^2 \equiv \widetilde{F}_4$ mod integers. Thus we must solve a system of mod 5 equations for a and b with augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -24 & -12 \\ 14 & 2 & 0 \\ 168 & 144 & 0 \end{array}\right).$$

The solution is $a = 1$, $b = 1/2$. We could also have used $b = 3$ since we are working mod 5.

Let $\widetilde{F}_4' = \widetilde{F}_4 - \frac{1}{10}\widetilde{F}_4^2 - \frac{1}{5}\widetilde{F}_8$. In grading 12, we have

$$\begin{aligned}
\widetilde{F}_4' &= -\tfrac{6}{25}m_{(12)} - \tfrac{12}{5}m_{(8,4)} - \tfrac{96}{5}m_{(6,2,2,2)} + \tfrac{12}{5}m_{(9,1,1,1)} + \tfrac{12}{25}m_{(5,5,1,1)} \\
\widetilde{F}_{12} &= m_{(12)} - 33m_{(8,4)} + 330m_{(4,4,4)} + 792m_{(6,2,2,2)} \\
\widetilde{F}_8\widetilde{F}_4 &= m_{12} + 15m_{(8,4)} + 42m_{(4,4,4)} + 168m_{(6,2,2,2)} - 12m_{(9,1,1,1)} - 168m_{(5,5,1,1)} - 2016m_{(3,3,3,3)} \\
\widetilde{F}_4^3 &= m_{(12)} + 3m_{(8,4)} + 6m_{(4,4,4)} + 432m_{(6,2,2,2)} - 36m_{(9,1,1,1)} - 72m_{(5,5,1,1)} - 1728m_{(3,3,3,3)}.
\end{aligned}$$

We wish to choose a, b, and c so that, in grading 12, $\frac{a}{25}\widetilde{F}_{12} + \frac{b}{25}\widetilde{F}_8\widetilde{F}_4 + \frac{c}{25}\widetilde{F}_4^3 \equiv \widetilde{F}_4'$ mod integers. Thus we must solve a system of equations mod 25 whose augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -6 \\ -33 & 15 & 3 & -60 \\ 330 & 42 & 6 & 0 \\ 792 & 168 & 432 & -480 \\ 0 & -12 & -36 & 60 \\ 0 & -168 & -72 & 12 \\ 0 & -2016 & -1728 & 0 \end{array}\right).$$

The solution is $a = 16$, $b = 7$, and $c = -4$.

We perform similar calculations for $\widetilde{F}_8$ in grading 12, then for $\widetilde{F}_4''$, $\widetilde{F}_8'$, and $\widetilde{F}_{12}$ in gradings 16 and then 20. ■

By the observation in the paragraph involving (4.5), the modified versions of F_4, F_8, and F_{12} given in Theorem 4.8, and also F_{20}, can be modified similarly in all subsequent gradings, yielding generators of the power series algebra $K^*(BX_{29}; \hat{\mathbb{Z}}_5)$ which we will call G_4, G_8, G_{12}, and G_{20}. By [24], $K^*(X_{29}; \hat{\mathbb{Z}}_5)$ is an exterior algebra on classes z_3, z_7, z_{11}, and z_{19} in $K^1(-)$ obtained using the map $e : \Sigma X = \Sigma\Omega BX \to BX$ and Bott periodicity $B : K^1(X) \to K^{-1}(X)$ by $z_i = B^{-1}e^*(G_{i+1})$. The following determination of the Adams operations is essential for our work on v_1-periodic homotopy groups. Here and elsewhere $QK^1(-)$ denotes the indecomposable quotient.

Theorem 4.9. *The Adams operation ψ^k in $QK^1(X_{29};\hat{\mathbb{Z}}_5)$ on the generators z_3, z_7, z_{11}, and z_{19} is given by the matrix*

$$\begin{pmatrix} k^3 & 0 & 0 & 0 \\ \frac{1}{5}k^3 - \frac{1}{5}k^7 & k^7 & 0 & 0 \\ \frac{24}{25}k^3 - \frac{8}{25}k^7 - \frac{16}{25}k^{11} & \frac{8}{5}k^7 - \frac{8}{5}k^{11} & k^{11} & 0 \\ \frac{92}{125}k^3 - \frac{12}{125}k^7 - \frac{16}{125}k^{11} - \frac{64}{125}k^{19} & \frac{12}{25}k^7 - \frac{8}{25}k^{11} - \frac{4}{25}k^{19} & \frac{1}{5}k^{11} - \frac{1}{5}k^{19} & k^{19} \end{pmatrix}.$$

Proof. We first note that

$$\psi^k(\ell_0(x)) = \ell_0(\psi^k x) = \ell_0((x+1)^k - 1) = \log((x+1)^k) = k\log(x+1) = k\ell_0(x).$$

Since F_{4j} is homogeneous of degree $4j$ in $\ell_0(x_i)$, $\psi^k(F_{4j}) = k^{4j}F_{4j}$. We can use this to determine ψ^k on the generators G_i which are defined as algebraic combinations of F_{4j}'s. We then apply e^* to this formula to obtain ψ^k in $K^{-1}(X_{29};\hat{\mathbb{Z}}_5)$. Since e^* annihilates decomposables, we need consider only the linear terms in the expressions which express G_i in terms of F_{4j}'s. On the basis (over $\hat{\mathbb{Q}}_5$) $\langle e^*(F_4), e^*(F_8), e^*(F_{12}), e^*(F_{20})\rangle$, the matrix of ψ^k is $D = \mathrm{diag}(k^4, k^8, k^{12}, k^{20})$. On the basis (over $\hat{\mathbb{Z}}_5$)

$$\langle e^*(G_4), e^*(G_8), e^*(G_{12}), e^*(G_{20})\rangle,$$

it is $P^{-1}DP$, where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{5} & 1 & 0 & 0 \\ -\frac{16}{25} & -\frac{8}{5} & 1 & 0 \\ -\frac{64}{125} & -\frac{4}{25} & -\frac{1}{5} & 1 \end{pmatrix}$$

is the change-of-basis matrix, obtained using the linear terms in 4.8. The matrix in the statement of the theorem is obtained by dividing $P^{-1}DP$ by k, since ψ^k in $K^1(-)$ corresponds to ψ^k/k in $K^{-1}(-)$. ∎

We can use Theorem 4.9 to obtain the v_1-periodic homotopy groups of $(X_{29})_5$ as follows.

Theorem 4.10. *The groups $v_1^{-1}\pi_*(X_{29})_{(5)}$ are given by*

$$v_1^{-1}\pi_{2t-1}(X_{29}) \approx v_1^{-1}\pi_{2t}(X_{29}) \approx \begin{cases} 0 & t \not\equiv 3 \quad (4) \\ \mathbb{Z}/5^3 & t \equiv 3, 15 \quad (20) \\ \mathbb{Z}/5^{\min(8, 3+\nu_5(t-7-4\cdot 5^4))} & t \equiv 7 \quad (20) \\ \mathbb{Z}/5^{\min(12, 3+\nu_5(t-11-4\cdot 5^8))} & t \equiv 11 \quad (20) \\ \mathbb{Z}/5^{\min(20, 3+\nu_5(t-19-12\cdot 5^{16}))} & t \equiv 19 \quad (20). \end{cases}$$

Proof. We use the result of [9] that $v_1^{-1}\pi_{2t}(X)_{(5)}$ is presented by the matrix $\begin{pmatrix} (\Psi^5)^T \\ (\Psi^2)^T - 2^t I \end{pmatrix}$. We form this matrix by letting $k = 5$ and 2 in the matrix of Theorem 4.9 and letting $x = 2^t$, obtaining

$$\begin{pmatrix} 125 & -15600 & -31274880 & -9765631257408 \\ 0 & 78125 & -78000000 & -3051773400000 \\ 0 & 0 & 48828125 & -3814687500000 \\ 0 & 0 & 0 & 19073486328125 \\ 8-x & -24 & -1344 & -268704 \\ 0 & 128-x & -3072 & -84480 \\ 0 & 0 & 2048-x & -104448 \\ 0 & 0 & 0 & 524288-x \end{pmatrix}. \tag{4.11}$$

Pivoting on the units (over $\mathbb{Z}_{(5)}$) in positions (5,2) and (7,4) and removing their rows and columns does not change the group presented. We now have a 6-by-2 matrix, whose nonzero entries are polynomials in x of degree 1 or 2. If $x \not\equiv 3$ mod 5, which is equivalent to $t \not\equiv 3$ mod 4, the bottom two rows are $\begin{pmatrix} u_1 & B \\ 0 & u_2 \end{pmatrix}$ with u_i units, and so the group presented is 0.

Henceforth, we assume $x \equiv 3$ mod 5. The polynomial in new position (5,2) is nonzero mod 5 for such x, and so we pivot on it, and remove its row and column. The five remaining entries are ratios of polynomials with denominator nonzero mod 5. Let $p_1, \ldots, p_5$ denote the polynomials in the numerators. The group $v_1^{-1}\pi_{2t}(X_{29})_{(5)}$ is $\mathbb{Z}/5^e$, where $e = \min(\nu(p_1(x)), \ldots, \nu(p_5(x)))$, where $x = 2^t$. We abbreviate $\nu_5(-)$

to $\nu(-)$ throughout the remainder of this section. We have

$$\begin{aligned}
p_1 &= -71122941747658752 + 9480741773824512x - 74067383851199x^2 + 33908441866x^3 \\
p_2 &= -66750692556800000 + 8897903174800000x - 69512640100000x^2 + 31789306250x^3 \\
p_3 &= -8327872 \cdot 10^{10} + 11101145 \cdot 10^9 x - 86731015625000x^2 + 39736328125x^3 \\
p_4 &= 4 \cdot 10^{19} - 533203125 \cdot 10^{10} x + 41656494140625000x^2 - 19073486328125x^3 \\
p_5 &= 1099511627776 - 146567856128x + 1145324544x^2 - 526472x^3 + x^4.
\end{aligned}$$

For values of m listed in the table, we compute and present in Table 4.13 the tuples (e_0, e_1, e_2, e_3) so that, up to units,

$$p_i(2^m + y) = 5^{e_0} + 5^{e_1} y + 5^{e_2} y^2 + 5^{e_3} y^3 \tag{4.12}$$

(plus y^4 if $i = 5$). Considerable preliminary calculation underlies the choice of these values of m.

Table 4.13. *Exponents of polynomials*

m	$i=1$	2	3	4	5
3	$3, 2, 1, 0$	$\infty, 7, 6, 5$	$\infty, 12, 12, 10$	$\infty, 21, 20, 19$	$\infty, 3, 2, 1$
15	$3, 2, 1, 0$	$8, 7, 6, 5$	$13, 12, 11, 10$	$22, 21, 20, 19$	$4, 4, 3, 2$
$7 + 4 \cdot 5^4$	$8, 2, 1, 0$	$8, 7, 6, 5$	$17, 12, 11, 10$	$26, 21, 21, 19$	$8, 3, 2, 1$
$11 + 4 \cdot 5^8$	$12, 2, 1, 0$	$12, 7, 7, 5$	$13, 12, 11, 10$	$\infty, 21, 20, 19$	$12, 3, 2, 1$
$19 + 12 \cdot 5^{16}$	$23, 2, 2, 0$	$20, 7, 6, 5$	$21, 12, 11, 10$	$22, 21, 20, 19$	$20, 3, 2, 1$

Recall that $\nu(2^{4 \cdot 5^i} - 1) = i + 1$, as is easily proved by induction. Thus

$$p(2^{m+20j}) = p(2^m + 2^m(2^{20j} - 1)) = p(2^m + 25j \cdot u), \tag{4.14}$$

with u a unit. Hence

$$\min\{\nu(p_i(2^{3+20j})) : 1 \le i \le 5\} = 3$$

since

$$p_1(2^{3+20j}) = p_1(2^3 + 5^2 ju) = 5^3 + 5^2 \cdot 5^2 ju + 5(5^2 ju)^2 + (5^2 ju)^3,$$

omitting some unit coefficients. Here we have set $y = 5^2 ju$ in (4.12). Replacing 3 by 15 yields an identical argument. This yields the second line of Theorem 4.10.

We use Table 4.13 to show

$$\min\{\nu(p_i(2^{19+12\cdot 5^{16}+20j})) : 1 \le i \le 5\} = \min(20, 4+\nu(j)) = \min(20, 3+\nu(20j)). \tag{4.15}$$

Indeed, for $\nu(j) \le 16$, the minimum is achieved when $i = 1$, with the 4 coming as $2+2$ with one 2 being from the 25 in (4.14) and the other 2 being the first 2 in the last row of Table 4.13. If $\nu(j) > 16$, the minimum is achieved when $i = 2$, using the first 20 in the last row of 4.13. The last case of Theorem 4.10 follows easily from (4.15), and the other two parts of 4.10 are obtained similarly.

To see that $v_1^{-1}\pi_{2t-1}(X_{29}) \approx v_1^{-1}\pi_{2t}(X_{29})$, we argue in three steps. First, the two groups have the same order using [9, 8.5] and the fact that the kernel and cokernel of an endomorphism of a finite group have equal orders. Second, by [16, 4.4], a presentation of $v_1^{-1}\pi_{2t-1}(X_{29})$ is given by $\begin{pmatrix} \Psi^5 \\ \Psi^2 - 2^t \end{pmatrix}$, i.e. like that for $v_1^{-1}\pi_{2t}(X_{29})$ except that the two submatrices are not transposed. Third, we pivot on this matrix, which is (4.11) with the top and bottom transposed, and find that we can pivot on units three times, so that the group presented is cyclic. ■

Next we provide the evidence for Conjecture 4.1 in the following result.

Proposition 4.16. *Let $B_0 = B(7, 15, 23, 39)$ denote the space constructed just before Conjecture 4.1. The Adams modules $K^*(X_{29}; \hat{\mathbb{Z}}_5)$ and $K^*(B_0; \hat{\mathbb{Z}}_5)$ are isomorphic.*

Proof. We use the 5-typical basis of $QK^1(SU(20))$ using powers of $y = e_p(\ell_0(x))$. Here $e_p(-)$ is the series inverse to $\ell_p(-)$, and x is the usual generator of $QK^1(SU(n))$. Then $\psi^k(y) = e_p(k\ell_p(y))$ only involves powers $y^{1+m(p-1)}$. See [32, pp.660-661] for a discussion of this. We compute ψ^2 on the basis $\langle y^3, y^7, y^{11}, y^{15}, y^{19}\rangle$ to be given by

$$\Psi^2 = \begin{pmatrix} 8 & 0 & 0 & 0 & 0 \\ -72 & 128 & 0 & 0 & 0 \\ 1368 & -2688 & 2048 & 0 & 0 \\ -32472 & 67200 & -67584 & 32768 & 0 \\ 865152 & -1841280 & 2095104 & -1474560 & 524288 \end{pmatrix}. \tag{4.17}$$

We find that $(0, 0, 0, 1, 3)^T$ is an eigenvector of Ψ^2 for $\lambda = 2^{31}$. We take the quotient by this vector, using $\langle y^3, y^7, y^{11}, y^{19}\rangle$ as the new basis. To obtain the matrix of ψ^2 on this basis, we subtract 3 times the fourth row of (4.17) from the fifth, and remove the fourth row and column, obtaining

$$\Psi_B^2 = \begin{pmatrix} 8 & 0 & 0 & 0 \\ -72 & 128 & 0 & 0 \\ 1368 & -2688 & 2048 & 0 \\ 962568 & -2042880 & 2297856 & 524288 \end{pmatrix}.$$

Let Ψ_X^2 denote the transpose of the bottom half of (4.11) without the $-x$ part. We desire an invertible matrix Q over $\mathbb{Z}_{(5)}$ such that $\Psi_X^2 Q = Q\Psi_B^2$. To find such a Q we find matrices Q_1 and Q_2 whose columns are eigenvectors of Ψ_X^2 and Ψ_B^2, respectively. These columns can be multiplied by any scalars. We can find scalars so that $Q := Q_1 Q_2^{-1}$ has entries in $\mathbb{Z}_{(5)}$ with units along the diagonal. We find that

$$Q = \begin{pmatrix} 51 & 0 & 0 & 0 \\ 21/2 & -1/2 & 0 & 0 \\ 48 & 2 & -2 & 0 \\ 30 & 16 & -18 & -4 \end{pmatrix}$$

works, and it also satisfies $\psi_X^k Q = Q\Psi_B^k$ for any k. ■

Since v_1-periodic homotopy groups can be computed from Adams modules, we also have $v_1^{-1}\pi_*(X_{29})_{(5)} \approx v_1^{-1}\pi_*(B_0)_{(5)}$.

For the cases discussed in Table 3.2, we could always construct equivalences from a factor $B(-,-,-)$ to $(X_n)_p$ because it was always the case that if the B-space had a d-cell, then $\pi_{d-1}((X_n)_p) = 0$. This will not be the case when trying to relate $B_0 = B(7, 15, 23, 39)$ and $(X_{29})_5$, primarily because $p = 5$ is so small compared to the dimensions of some of the cells in B_0. For example, localized at $p = 5$, $\pi_{37}(S^7) \approx \mathbb{Z}/5$, the first unstable class. See, e.g., Diagram 3.4 or [5, 5.16]. This gives a nonzero element in $\pi_{37}(X_{29})$ which provides a possible obstruction to extending a map from the 37-skeleton of B_0 over the 38-cell representing the class $x_{15}x_{23}$. A similar problem occurs due to $\pi_{45}(S^7) \to \pi_{45}(X_{29})$ being nonzero, giving a possible obstruction to extending over x_7x_{39}. Also, $\alpha_1\beta_1$ on S^{23} gives an apparent element in $\pi_{68}(X_{29})$ which is a possible obstruction to extending over $x_7x_{23}x_{39}$. It is conceivable that more delicate arguments such as that on [29, p.661] might show that these obstructions can be removed. By [9, 3.4,8.1] and our Proposition 4.16, the associated v_1-periodic spectra ΦB_0 and ΦX_{29} are equivalent. However, the fact that, as we shall see at the end of this section, there is not a space related to $SU(N)$ which might be equivalent to $(X_{31})_5$ tempers a belief that there should be a general reason for B_0 and $(X_{29})_5$ to be equivalent.

Note that the entry in position $(4,3)$ of the matrix of 4.9 implies that the 39-cell of $(X_{29})_5$ is attached to the 23-cell by α_2, which is not detected by primary Steenrod operations. Note also that the spherical resolvability of $(X_{29})_5$ and also the spaces $(X_{31})_5$ and $(X_{34})_7$ follows from [15].

We can determine the Adams operations and v_1-periodic homotopy groups of $(X_{31})_5$ by an argument very similar to that used above for $(X_{29})_5$. We shall merely sketch. The analogue of Theorem 4.2 is

Theorem 4.18. *There is an isomorphism $H^*(BX_{31};\hat{\mathbb{Z}}_5) \approx H^*(BT;\hat{\mathbb{Z}}_5)^{G_{31}}$, where G_{31} has the four generators given for G_{29} in the proof of 4.2 and also $\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&1 \end{pmatrix}$. Then $H^*(BT;\hat{\mathbb{Z}}_5)^{G_{31}}$ is a polynomial ring on the generators f_8, f_{12}, and f_{20} given in 4.2 together with*

$$\begin{aligned} f_{24} \;=\; & m_{(24)} - 66m_{(20,4)} + 1023m_{(16,8)} + 2180m_{(12,12)} + 1293156m_{(8,8,4,4)} \\ & +267096m_{(12,4,4,4)} + 2121984m_{(6,6,6,6)} + 620352m_{(10,6,6,2)} - 4032m_{(14,6,2,2)} \\ & -190080m_{(10,10,2,2)} - 11892m_{(12,8,4)} - 4938m_{(16,4,4)} - 24534m_{(8,8,8)} \\ & -2304m_{(18,2,2,2)}. \end{aligned}$$

The analogue of Theorem 4.8 is

Theorem 4.19. *Let $f_8, f_{12}, f_{20}, f_{24}$ be as in 4.18, and let*

$$F_j = f_j(\ell_0(x_1),\dots,\ell_0(x_4)).$$

Then the following series are 5-integral through grading 24.

$$\begin{gathered} F_8 - \tfrac{8}{5}F_{12} - \tfrac{7}{25}F_8^2 - \tfrac{4}{25}F_{20} - \tfrac{21}{125}F_8F_{12} - \tfrac{99}{125}F_{24} - \tfrac{597}{625}F_8^3 - \tfrac{558}{625}F_{12}^2 \\ F_{12} - \tfrac{2}{5}F_8^2 - \tfrac{1}{5}F_{20} - \tfrac{4}{25}F_8F_{12} - \tfrac{18}{25}F_{24} - \tfrac{74}{125}F_8^3 - \tfrac{11}{125}F_{12}^2 \\ F_{20} - \tfrac{3}{5}F_{24} - \tfrac{2}{5}F_8^3. \end{gathered}$$

The analogue of 4.9 is

Theorem 4.20. *The Adams operation ψ^k in $K^1(X_{31};\hat{\mathbb{Z}}_5)$ on the generators z_7, z_{11}, z_{19}, and z_{23} is given by the matrix*

$$\begin{pmatrix} k^7 & 0 & 0 & 0 \\ \frac{8}{5}k^7 - \frac{8}{5}k^{11} & k^{11} & 0 & 0 \\ \frac{12}{25}k^7 - \frac{8}{25}k^{11} - \frac{4}{25}k^{19} & \frac{1}{5}k^{11} - \frac{1}{5}k^{19} & k^{19} & 0 \\ \frac{279}{125}k^7 - \frac{168}{125}k^{11} - \frac{12}{125}k^{19} - \frac{99}{125}k^{23} & \frac{21}{25}k^{11} - \frac{3}{25}k^{19} - \frac{18}{25}k^{23} & \frac{3}{5}k^{19} - \frac{3}{5}k^{23} & k^{23} \end{pmatrix}.$$

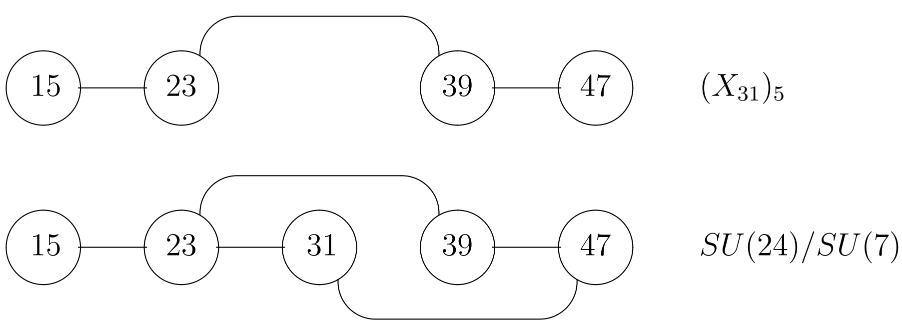
The analogue of Theorem 4.10 is

Theorem 4.21. *The groups $v_1^{-1}\pi_*(X_{31})_{(5)}$ are given by*

$$v_1^{-1}\pi_{2t-1}(X_{31}) \approx v_1^{-1}\pi_{2t}(X_{31}) \approx \begin{cases} 0 & t \not\equiv 3 \quad (4) \\ \mathbb{Z}/5^3 & t \equiv 7, 15 \quad (20) \\ \mathbb{Z}/5^{\min(8, 3+\nu_5(t-11-8\cdot 5^4))} & t \equiv 11 \quad (20) \\ \mathbb{Z}/5^{\min(12, 3+\nu_5(t-19-16\cdot 5^8))} & t \equiv 19 \quad (20) \\ \mathbb{Z}/5^{\min(20, 3+\nu_5(t-23-16\cdot 5^{16}))} & t \equiv 23 \quad (20). \end{cases}$$

One might hope that, analogously to 4.1, there might be a space $B(15, 23, 39, 47)$ related to $SU(24)$ and equivalent to $(X_{31})_5$. This cannot happen. Diagram 4.22, in which straight lines denote α_1 attaching maps and curved lines α_2, shows the ways in which the generating cells of $(X_{31})_5$ and one factor of $SU(24)/SU(7)$ are attached to one another. Because the 31-cell is attached to a lower cell and has a higher cell attached to it, there cannot be a map in either direction between these spaces sending generators across. This is what puts a damper on any hope that all p-compact groups are related to unitary groups.

Diagram 4.22. *Attaching maps of generating cells*



5. The 7-primary modular case

In this section, we first give in Theorem 5.1 new explicit formulas for the six polynomials which generate as a polynomial algebra the invariant ring of the complex reflection group G_{34} of [33], called the Mitchell group in [14]. Over $\hat{\mathbb{Z}}_7$, the invariant ring of G_{34} is also a polynomial algebra, but the generators must be altered slightly from the complex case, as we show prior to 5.6. Next we use this information to find explicit generators for $K^*(BX_{34};\hat{\mathbb{Z}}_7)$ in 5.6, and from this the Adams operations in $QK^1(X_{34};\hat{\mathbb{Z}}_7)$ in 5.15. These in turn enable us to compute the v_1-periodic homotopy groups $v_1^{-1}\pi_*(X_{34})_{(7)}$. Finally, we show in 5.18 that the Adams module $QK^1(X_{34};\hat{\mathbb{Z}}_7)$ is isomorphic to that of a space formed from $SU(42)$; similarly to 4.1, we conjecture in 5.17 that this isomorphism is induced by a homotopy equivalence.

Theorem 5.1. *The complex invariants of the reflection group G_{34} (defined in the proof) form a polynomial algebra*

$$\mathbb{C}[x_1,\ldots,x_6]^{G_{34}} \approx \mathbb{C}[f_6, f_{12}, f_{18}, f_{24}, f_{30}, f_{42}]$$

with generators given by

$$f_{6k} = (1+(-1)^k 27^{k-1}\cdot 5)m_{(6k)} + \sum_{s=1}^{k}\binom{6k}{3s}(1+(-1)^{k+s}27^{k-1})m_{(6k-3s,3s)} + \sum_{\mathbf{e}}(\mathbf{e})m_{\mathbf{e}},$$

where $\mathbf{e}$ ranges over all partitions $\mathbf{e}=(e_1,\ldots,e_r)$ of $6k$ with $3\le r\le 6$ satisfying $e_i \equiv e_j$ mod 3 for all i,j, and $e_i\equiv 0$ mod 3 if $r<6$. Here also $(\mathbf{e})$ denotes the multinomial coefficient $(e_1+\cdots+e_r)!/(e_1!\cdots e_r!)$, and $m_{\mathbf{e}}$ the monomial symmetric polynomial, which is the shortest symmetric polynomial in $x_1,\ldots,x_6$ containing $x_1^{e_1}\cdots x_r^{e_r}$.

For example, we have

- $f_6 = -4m_{(6)} + 40m_{3,3} + 720m_{(1,1,1,1,1,1)}$;
- $f_{12} = 136m_{(12)} - 26\binom{12}{3}m_{(9,3)} + 28\binom{12}{6}m_{(6,6)} + \sum(\mathbf{e})m_{\mathbf{e}}$, where $\mathbf{e}$ ranges over

$$\{(6,3,3),\ (3,3,3,3),\ (2,2,2,2,2,2),\ (7,1,1,1,1,1),\ (4,4,1,1,1,1)\}.$$

- $f_{18} = (1-5{\cdot}27^2)m_{(18)}+\binom{18}{3}(1+27^2)m_{(15,3)}+\binom{18}{6}(1-27^2)m_{(12,6)}+ \binom{18}{9}(1+27^2)m_{(9,9)} + \sum(\mathbf{e})m_{\mathbf{e}}$, where $\mathbf{e}$ ranges over

$$\{(12,3,3),\ (9,6,3),\ (9,3,3,3),\ (6,6,6),\ (6,6,3,3),\ (6,3,3,3,3),\ (3,3,3,3,3,3),$$
$$(13,1,1,1,1,1),\ (10,4,1,1,1,1),\ (7,4,4,1,1,1),\ (4,4,4,4,1,1),\ (7,7,1,1,1,1),$$
$$(8,2,2,2,2,2),\ (5,5,2,2,2,2)\}$$

Proof of Theorem 5.1. As described in [33], the reflection group G_{34} is generated by reflections across the following hyperplanes in $\mathbb{C}^6$: $x_i - x_j = 0$, $x_1 - \omega x_2 = 0$, and $x_1+x_2+x_3+x_4+x_5+x_6 = 0$. Here $\omega = e^{2\pi i/3}$. It follows easily that G_{34} is generated by all permutation matrices together with the following two:

$$\begin{pmatrix} 0 & \omega^2 & 0 & 0 & 0 & 0 \\ \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad I - \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{5.2}$$

In [14], Conway and Sloane consider G_{34} instead as the automorphisms of a certain $\mathbb{Z}[\omega]$-lattice in $\mathbb{C}^6$. The lattice has 756 vectors of norm 2. There are none of smaller positive norm. 270 of these vectors are those with ω^a in one position, $-\omega^b$ in another, and 0 in the rest. Here, of course, a and b can be 0, 1, or 2. The other 486 are those of the form $\pm\frac{1}{\sqrt{-3}}(\omega^{a_1},\ldots,\omega^{a_6})$ such that $\sum a_i \equiv 0 \bmod 3$.

As a partial verification that this lattice approach to G_{34} is consistent with the reflection approach, one can verify that the reflection matrices permute these 756 vectors. It is obvious that permutation matrices do, and easily verified for the first matrix of (5.2). The second matrix of (5.2), which has order 2, sends

- $(\omega,\omega^2,0,0,0,0)$ to $\frac{1}{\sqrt{-3}}(\omega^2,\omega,1,1,1,1)$;
- $\frac{1}{\sqrt{-3}}(1,1,1,1,1,1)$ to $-\frac{1}{\sqrt{-3}}(1,1,1,1,1,1)$;
- $\frac{1}{\sqrt{-3}}(1,1,1,\omega,\omega,\omega)$ to $-\frac{1}{\sqrt{-3}}(\omega,\omega,\omega,1,1,1)$;
- $\frac{1}{\sqrt{-3}}(1,1,\omega,\omega,\omega^2,\omega^2)$ to itself.

After permutation, negation, and multiplication by ω, this takes care of virtually all cases.

Let

$$p_m(x_1,\dots,x_6)=\sum_{(v_1,\dots,v_6)}(v_1x_1+\cdots+v_6x_6)^m, \tag{5.3}$$

where the sum is taken over the 756 vectors described above. Then p_m is invariant under G_{34} for every positive integer m. It is proved in [14, Thm.10] that the ring of complex invariant polynomials is given by

$$\mathbb{C}[x_1,\dots,x_6]^{G_{34}}=\mathbb{C}[p_6,p_{12},p_{18},p_{24},p_{30},p_{42}]. \tag{5.4}$$

In [14], several other lattices isomorphic to the above one are described, any of which can be used to give a different set of vectors v and invariant polynomials p_m, still satisfying (5.4). The one that we have selected seems to give the simplest polynomials; in particular, the only ones with integer coefficients.

We have $p_{6k}=S_1+S_2$, where $S_1=\sum_{i\ne j}\sum_{a,b=0}^2(\omega^a x_i-\omega^b x_j)^{6k}$, with $1\le i,j\le 6$, and

$$S_2=\frac{2}{(-3)^{3k}}\sum_{a_i=0}^{2}(\omega^{a_1}x_1+\cdots+\omega^{a_5}x_5+\omega^{-a_1-\cdots-a_5}x_6)^{6k}.$$

The coefficient of 2 on S_2 is due to the ±1. Note that the sum for S_1 has $6\cdot5\cdot3^2$ terms, while that for S_2 has 3^5 terms. Next note that if a term T^{6k} occurs in either sum, then so does $(\omega T)^{6k}$ and $(\omega^2T)^{6k}$, and all are equal. Thus we obtain $S_1=3\sum_{i\ne j}\sum_{b=0}^2(x_i-\omega^b x_j)^{6k}$ and

$$S_2=3\frac{2}{(-3)^{3k}}\sum_{a_2,\dots,a_5=0}^{2}(x_1+\omega^{a_2}x_2+\cdots+\omega^{a_5}x_5+\omega^{-a_2-\cdots-a_5}x_6)^{6k}.$$

We simplify S_1 further as

$$\begin{aligned} S_1 &= 3\sum_{\ell=0}^{6k}(-1)^\ell\tbinom{6k}{\ell}\sum_{i\ne j}x_i^\ell x_j^{6k-\ell}\sum_{b=0}^{2}\omega^{b\ell}\\ &= 9\sum_{s=0}^{2k}(-1)^s\tbinom{6k}{3s}\sum_{i\ne j}x_i^{3s}x_j^{6k-3s}\\ &= 18\Big(5m_{6k}+\sum_{s=1}^{k}(-1)^s\tbinom{6k}{3s}m_{(6k-3s,3s)}\Big). \end{aligned}$$

At the first step, we have used that $\sum_{b=0}^2\omega^{b\ell}$ equals 0 if $\ell\not\equiv 0$ mod 3, and equals 3 if $\ell\equiv 0$ mod 3. At the second step, we have noted that $\sum_{i\ne j}x_i^{3s}x_j^{6k-3s}$ equals $m_{(6k-3s,3s)}$ if $s\notin\{0,k,2k\}$, it equals $2m_{(3k,3k)}$ if $s=k$, and equals $5m_{(6k)}$ if $s=0$ or $2k$.

The sum S_2 becomes

$$\begin{aligned} S_2 &= \frac{6}{(-3)^{3k}}\sum_{\mathbf{e}}(\mathbf{e})\sum_{a_2=0}^{2}(\omega^{e_2-e_6})^{a_2}\cdots\sum_{a_5=0}^{2}(\omega^{e_5-e_6})^{a_5}x_1^{e_1}\cdots x_6^{e_6} \\ &= \frac{6}{(-27)^k}\sum_{e_1\equiv\cdots\equiv e_6\ (3)}(\mathbf{e})3^4x_1^{e_1}\cdots x_6^{e_6}. \end{aligned}$$

Then $(-27)^k(S_1+S_2)/486$ equals the expression which we have listed for f_{6k} in the statement of the theorem. We have chosen to work with this rather than p_{6k} itself for numerical simplicity. It is important that the omitted coefficient is not a multiple of 7.

For good measure, we show that (5.3) is 0 if $m \not\equiv 0$ (6). If $m \not\equiv 0$ mod 3, then replacing terms T^m by $(\omega T)^m$ leaves the sums like S_1 and S_2 for (5.4) unchanged while, from a different perspective, it multiplies them by ω^m. Thus the sums are 0. If $m \equiv 3$ mod 6, the term in S_1 corresponding to $\sum x_i^{3s}x_j^{m-3s}$ occurs with opposite sign to that corresponding to $\sum x_i^{m-3s}x_j^s$, and so $S_1 = 0$. For S_2, the $(\pm 1)^m$ will cause pairs of terms to cancel. ■

Remark 5.5. The only other place known to the author where formulas other than (5.3) for these polynomials exist is [26], where they occupy 190 pages of dense text when printed.

As pointed out by Kasper Andersen, $f_{42}-(f_6)^7$ is divisible by 7. This is easily seen by expanding $(f_6)^7 = (\sum(v_1x_1+\cdots+v_6x_6)^6)^7$ by the multinomial theorem. The need for this became apparent to Andersen, as the author had thought that the invariant ring of G_{34} over $\hat{\mathbb{Z}}_7$ was $\hat{\mathbb{Z}}_7[f_6,\ldots,f_{42}]$, and this would have led to an impossible conclusion for the Adams operations in $QK^1(X_{34};\hat{\mathbb{Z}}_7)$.

Let $h_{42} = \frac{1}{7}(f_{42}-(f_6)^7)$. Then we have the following result, for which we are grateful to Andersen.

Theorem 5.6. *The invariant ring of G_{34} over $\hat{\mathbb{Z}}_7$ is given by*

$$\hat{\mathbb{Z}}_7[x_1,\ldots,x_6]^{G_{34}} = \hat{\mathbb{Z}}_7[f_6,f_{12},f_{18},f_{24},f_{30},h_{42}].$$

Proof. A `Magma` program written and run by Andersen showed that each of these asserted generators is indecomposable over $\mathbb{Z}/7$. (This is what failed when f_{42} was used; it equals $(f_6)^7$ over $\mathbb{Z}/7$.) Thus the result follows from (5.4). ■

Since f_{36} is invariant under G_{34}, it follows from (5.4) that it can be decomposed over $\mathbb{C}$ in terms of f_6, f_{12}, f_{18}, f_{24}, and f_{30}. The nature of the coefficients in this decomposition was not so clear. It turned out that all coefficients were rational numbers which are 7-adic units. We make this precise in

Theorem 5.7. *f_{36} can be decomposed as*

$$\begin{aligned} &q_1 f_6 f_{30} + q_2 f_{12} f_{24} + q_3 f_{18}^2 + q_4 f_6^2 f_{24} + q_5 f_6 f_{12} f_{18} \\ &+ q_6 f_{12}^3 + q_7 f_6^3 f_{18} + q_8 f_6^2 f_{12}^2 + q_9 f_6^4 f_{12} + q_{10} f_6^6 \end{aligned}$$

with

$$\begin{aligned} q_1 &= 944610925401/15161583716 \\ q_2 &= 733671261/19519520 \\ q_3 &= 243068633/9781739 \\ q_4 &= -133840666859131062549/73986709144034080 \\ q_5 &= -1758887990521258018071215403/629320589839873719708800 \\ q_6 &= -1602221942044323/4879880000000 \\ q_7 &= 4011206338081535787030788541/114421925425431585401600 \\ q_8 &= 701461342458322269763709951654931/15733014745996842992720000000 \\ q_9 &= -11844219519446025955021712628669/22348032309654606523750000 \\ q_{10} &= 26589469730264682368719198549833/22348032309654606523750000 \end{aligned}$$

Each of these coefficients q_i is a 7-adic unit; i.e. no numerator or denominator is divisible by 7.

Proof. The ten products, $f_6 f_{30}, \ldots, f_6^6$, listed above are the only ones possible. We express each of these products as a combination of monomial symmetric polynomials $m_{\mathbf{e}}$. We use `Magma` to do this. The length of $m_{(e_1,\ldots,e_r)}$ is defined to be r. We only kept track of components of the products of length ≤ 4. This meant that we only had to include components of length ≤ 4 of the various f_{6k} being multiplied.

There were 34 $m_{\mathbf{e}}$'s of length ≤ 4. These correspond to the partitions of 36 into multiples of 3. (Note that monomials with subscripts $\equiv 1$ or 2 mod 3 only occur for us if the length is 6. Not having to deal with them simplifies our work considerably.) Indeed, there was one of length 1, six of length 2, twelve of length 3, and fifteen of length 4. `Magma` expressed each monomial such as $f_6 f_{30}$ or f_6^6 as an integer combination of these, plus monomials of greater length. We just ignored in the output all those of greater length. The coefficients in these expressions were typically 12 to 15 digits. We also wrote f_{36} as a combination of monomial symmetric polynomials of length ≤ 4, ignoring the longer ones. This did not require any fancy software, just the multinomial coefficients from Theorem 5.1.

Now we have a linear system of 34 linear equations with integer coefficients in 10 unknowns. The unknowns are the coefficients q_i in the equation at the beginning of 5.7, and the equations are the component monomials of length ≤ 4. Miraculously, there was a unique rational solution, as given in the statement of this theorem.

If it were not for the fact that the Conway-Sloane theorem 5.4 guarantees that there must be a solution when all monomial components (of length ≤ 6) are considered, then we would have to consider them all, but the fact that we got a unique solution looking at only the monomial components of length ≤ 4 implies that this solution will continue to hold in the other unexamined components. ■

Next we wish to modify the generators in 5.6 to obtain generators of $QK^1(X_{34}; \hat{\mathbb{Z}}_7)$. Similarly to 4.8, we let $\ell_0(x) = \ln(1+x)$, and

$$F_i = F_i(x_1, \dots, x_6) = f_i(\ell_0(x_1), \dots, f_6(\ell_0(x_6))). \tag{5.8}$$

A major calculation is required to modify the classes F_i so that their coefficients are in $\hat{\mathbb{Z}}_7$; i.e. they do not have 7's in the denominators. As observed after (4.5), it will be enough to accomplish this through grading 42 (with grading of x_i considered to be 1).

Theorem 5.9. *The following expressions are 7-integral through grading 42:*

- $F_{30} + \frac{5}{7}F_{36} + \frac{22}{7^2}F_{42}$;
- $F_{24} + \frac{4}{7}F_{30} + \frac{45}{7^2}F_{36} + \frac{104}{7^3}F_{42}$;
- $F_{18} + \frac{3}{7}F_{24} + \frac{20}{7^2}F_{30} + \frac{157}{7^3}F_{36} + \frac{526}{7^4}F_{42}$;
- $F_{12} + \frac{2}{7}F_{18} + \frac{45}{7^2}F_{24} + \frac{109}{7^3}F_{30} + \frac{1391}{7^4}F_{36} + \frac{6201}{7^5}F_{42}$;
- $F_6 + \frac{1}{7}F_{12} + \frac{22}{7^2}F_{18} + \frac{204}{7^3}F_{24} + \frac{1107}{7^4}F_{30} + \frac{9682}{7^5}F_{36} + \frac{100682}{7^6}F_{42}$.

It was very surprising that just linear terms were needed here. Decomposable terms were certainly expected. The analogue for G_{29} in 4.8 involved many decomposables. It would be interesting to know why Theorem 5.9 works with just linear terms; presumably this pattern will continue into higher gradings.

Proof of Theorem 5.9. Similarly to the proof of 4.8, we define

$$\widetilde{F}_i = \widetilde{F}_i(x_1, \dots, x_6) = f_i(\ell_p(x_1), \dots, \ell_p(x_6)),$$

and observe that a polynomial in the $\widetilde{F}_i$'s is 7-integral if and only if the same polynomial in the F_i's is.

Next note that in the range of concern for Theorem 5.9 $\ell_7(x) = x + x^7/7$. If we define

$$h_i = h_i(x_1, \dots, x_6) = f_i(x_1 + x_1^7, \dots, x_6 + x_6^7),$$

then 5.9 is clearly equivalent to

Statement 5.10. *For $t \geq 1$ and grading ≤ 42,*

- $h_{30} + 5h_{36} + 22h_{42} \equiv 0$ *mod* 7^t *in grading* $30 + 6t$*;*
- $h_{24} + 4h_{30} + 45h_{36} + 104h_{42} \equiv 0$ *mod* 7^t *in grading* $24 + 6t$*;*
- $h_{18} + 3h_{24} + 20h_{30} + 157h_{36} + 526h_{42} \equiv 0$ *mod* 7^t *in grading* $18 + 6t$*;*
- $h_{12} + 2h_{18} + 45h_{24} + 109h_{30} + 1391h_{36} + 6201h_{42} \equiv 0$ *mod* 7^t *in grading* $12 + 6t$*;*
- $h_6 + h_{12} + 22h_{18} + 204h_{24} + 1107h_{30} + 9682h_{36} + 100682h_{42} \equiv 0$ *mod* 7^t *in grading* $6 + 6t$.

We use `Maple` to verify 5.10. Our f_i's are given in Theorem 5.1 in terms of $m_{\mathbf{e}}$'s. To evaluate $m_{\mathbf{e}}(x_1+x_1^7, \dots, x_6+x_6^7)$, the following result keeps the calculation manageable (e.g. it does not involve a sum over all permutations). Partitions can be written either in increasing order or decreasing order; we use increasing. If $(a_1, \dots, a_r)$ is an r-tuple of positive integers, let $s(a_1, \dots, a_r)$ denote the sorted form of the tuple; i.e. the rearranged version of the tuple so as to be in increasing order. For example, $s(4, 2, 3, 2) = (2, 2, 3, 4)$.

Proposition 5.11. *The component of* $m_{(e_1,\dots,e_r)}(x_1+x_1^7,\dots,x_6+x_6^7)$ *in grading* $\sum e_i + 6t$ *is*

$$\sum_{\mathbf{j}} \frac{P(e_1+6j_1,\dots,e_r+6j_r)}{P(e_1,\dots,e_r)}\binom{e_1}{j_1}\cdots\binom{e_r}{j_r} m_{s(e_1+6j_1,\dots,e_r+6j_r)},$$

where $\mathbf{j}=(j_1,\dots,j_r)$ *ranges over all* r*-tuples of nonnegative integers summing to* t*, and* $P(a_1,\dots,a_r)$ *is the product of the factorials of repetend sizes.*

For example, $P(4,2,3,3)=2!$ because there are two 3's, $P(3,1,3,3,1,2)=3!2!$, and $P(3,4,2,1)=1$.

Example 5.12. *We consider as a typical example, the component of*

$$m_{(3,3,9,15)}(x_1+x_1^7,\dots,x_6+x_6^7)$$

in grading 42. Table 5.13 lists the possible values of $\mathbf{j}$ *and the contribution to the sum. The final answer is the sum of everything in the right hand column.*

Table 5.13. *Terms for Example 5.12*

$\mathbf{j}$	*term*
$(2,0,0,0)$	$\binom{3}{2}m_{3,9,15,15}$
$(0,2,0,0)$	$\binom{3}{2}m_{3,9,15,15}$
$(0,0,2,0)$	$\binom{9}{2}m_{3,3,15,21}$
$(0,0,0,2)$	$\binom{15}{2}m_{3,3,9,27}$
$(1,1,0,0)$	$3\cdot 3\cdot 3m_{9,9,9,15}$
$(1,0,1,0)$	$3\cdot 9m_{3,9,15,15}$
$(1,0,0,1)$	$3\cdot 15m_{3,9,9,21}$
$(0,1,1,0)$	$3\cdot 9m_{3,9,15,15}$
$(0,1,0,1)$	$3\cdot 15m_{3,9,9,21}$
$(0,0,1,1)$	$9\cdot 15m_{3,3,15,21}$

Proof of Proposition 5.11. $m_{(e_1,\dots,e_r)}(x_1+x_1^7,\dots,x_6+x_6^7)$ is related to

$$\sum_{\sigma}(x_{\sigma(1)}^{e_1}+\tbinom{e_1}{1}x_{\sigma(1)}^{e_1+6}+\tbinom{e_1}{2}x_{\sigma(1)}^{e_1+12}+\cdots)\cdots(x_{\sigma(r)}^{e_r}+\tbinom{e_r}{1}x_{\sigma(r)}^{e_r+6}+\cdots) \tag{5.14}$$

summed over all permutations σ in Σ_r. If t values of e_i are equal, then (5.14) will give $t!$ times the correct answer. That is the reason that we divide by $P(\mathbf{e})$. If $(e_1+6j_1,\dots,e_r+6j_r)$ contains s equal numbers, then the associated m will be obtained from each of $s!$ permutations, which is the reason that $P(e_1+6j_1,\dots,e_r+6j_r)$ appears in the numerator. ∎

At first, mimicking 4.8, we were allowing for products of h's in addition to the linear terms which appear in 5.10, but it was turning out that what was needed to satisfy the congruences was just the linear term. If just a linear term was going to work, the coefficients could be obtained by just looking at monomials of length 1. They were computed by `Maple`, using that, by 5.1 and 5.11, the coefficient of $m_{(6k+6t)}$ in h_{6k} is $(1+(-1)^k27^{k-1}\cdot 5)\binom{6k}{t}$. Write the kth expression from the bottom of 5.10 as $\sum_{j\ge 0} a_{j,k}h_{6k+6j}$. We require that the coefficient of $m_{(6k+6t)}$ in $\sum_{j=0}^{t} a_{j,k}h_{6k+6j}$ is 0 mod 7^t. But this coefficient equals

$$\sum_{j=0}^{t} a_{j,k}\tbinom{6k+6j}{t-j}(1+(-1)^{k+j}27^{k+j-1}\cdot 5).$$

We solve iteratively for $a_{j,k}$, starting with $a_{0,k}=1$, and obtain the values in 5.10. Note that it first gives $a_{1,1}\equiv 1 \bmod 7$. If we had chosen a value such as 8 or -6 instead of 1, then the value of $a_{2,1}$ would be different than 22. So these numbers $a_{j,k}$ are not uniquely determined. These different choices just amount to choosing a different basis for $QK^1(X_{34};\hat{\mathbb{Z}}_7)$.

Verifying Statement 5.10 required running many `Maple` programs. For each line of 5.10, a verification had to be made for each relevant t-value, from two t-values for the first line down to six t-values for the last line. Moreover, for each of these pairs (line number, t-value), it was convenient to use a separate program for monomials of each length 2, 3, 4, and 5, and then, for monomials of length 6, it was done separately for those with subscripts congruent to 0, 1, or 2 mod 3. Thus altogether $(2+3+4+5+6)(4+3)=140$ `Maple` programs were run. The programs had enough similarity that one could be morphed into another quite easily, and a more skillful programmer could incorporate them all into the same program.

Note that expanding from f_j to h_j does not change the number of components in monomials, nor does it change the mod 3 value of the sum of the subscripts (i.e. exponents) in the monomials. This is simpler than the situation in the proof of 5.7. The algorithm is quite easy. For each combination of h's in 5.10, replace each h_{6j} by the combination of $m_{\mathbf{e}}$'s in f_{6j} in 5.1, but expanded using 5.11. ∎

To obtain the Adams operations in $QK^1(X_{34};\hat{\mathbb{Z}}_7)$, we argue similarly to the paragraph which precedes Theorem 4.9. First note that F_{36} decomposes in terms of F_{6i}'s exactly as does f_{36} in terms of f_{6i}'s in 5.7. We can[7] modify by decomposables in dimensions greater than 42 to obtain 7-integral classes G_6, G_{12}, G_{18}, G_{24}, and G_{30} which agree with the classes of 5.9 (with F_{36} decomposed) through dimension 42. There is also a 7-integral class G_{42} which agrees with $\frac{1}{7}(F_{42}-(F_6)^7)$ in dimension 42. These generate $K^*(BX_{34};\hat{\mathbb{Z}}_7)$ as a power series algebra. As in the preamble to 4.9, then $z_i := B^{-1}e^*(G_{i+1})$ for $i = 5$, 11, 17, 23, 29, and 41 form a basis for $QK^1(X_{34};\hat{\mathbb{Z}}_7)$, and e^* annihilates decomposables.

Similarly to the situation for $(X_{29})_5$ in the proof of 4.9, if we let

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{7} & 1 & 0 & 0 & 0 & 0 \\ \frac{22}{49} & \frac{2}{7} & 1 & 0 & 0 & 0 \\ \frac{204}{343} & \frac{45}{49} & \frac{3}{7} & 1 & 0 & 0 \\ \frac{1107}{2401} & \frac{109}{343} & \frac{20}{49} & \frac{4}{7} & 1 & 0 \\ \frac{16647}{16807} & \frac{1399}{2401} & \frac{183}{343} & \frac{6}{49} & \frac{1}{7} & 1 \end{pmatrix},$$

then the matrix of ψ^k on the basis $\{z_5, z_{11}, z_{17}, z_{23}, z_{29}, z_{41}\}$ is

$$P^{-1}\mathrm{diag}(k^5, k^{11}, k^{17}, k^{23}, k^{29}, k^{41})P.$$

The entries in the last row of P are 7 times the coefficients of F_{42} in 5.9 reduced mod 1. Those coefficients were multiplied by 7 because z_{41} is related to $\frac{1}{7}F_{42}$ rather than to F_{42}.

Using this, we compute the v_1-periodic homotopy groups, similarly to 4.10. Note the remarkable similarity with that result. Here, of course, $\nu(-)$ denotes the exponent of 7 in an integer.

[7] But we need not bother to do so explicitly.

Theorem 5.15. *The groups $v_1^{-1}\pi_*(X_{34})_{(7)}$ are given by*

$$v_1^{-1}\pi_{2t-1}(X_{34}) \approx v_1^{-1}\pi_{2t}(X_{34}) \approx \begin{cases} 0 & t \not\equiv 5\ (6) \\ \mathbb{Z}/7^5 & t \equiv 5, 35\ (42) \\ \mathbb{Z}/7^{\min(12, 5+\nu(t-11-12\cdot 7^6))} & t \equiv 11\ (42) \\ \mathbb{Z}/7^{\min(18, 5+\nu(t-17-18\cdot 7^{12}))} & t \equiv 17\ (42) \\ \mathbb{Z}/7^{\min(24, 5+\nu(t-23-18\cdot 7^{18}))} & t \equiv 23\ (42) \\ \mathbb{Z}/7^{\min(30, 5+\nu(t-29-12\cdot 7^{24}))} & t \equiv 29\ (42) \\ \mathbb{Z}/7^{\min(42, 5+\nu(t-41-24\cdot 7^{36}))} & t \equiv 41\ (42). \end{cases}$$

Proof. The group $v_1^{-1}\pi_{2t}(X)_{(7)}$ is presented by $\begin{pmatrix} (\psi^7)^T \\ (\psi^3)^T - 3^t I \end{pmatrix}$, since 3 generates $\mathbb{Z}/49^\times$. We let $x = 3^t$ and form this matrix analogously to (4.11). Five times we can pivot on units, removing their rows and columns, leaving a column matrix with 7 polynomials in x. The 7-exponent of $v_1^{-1}\pi_{2t}(X)_{(7)}$ is the smallest of that of these polynomials (with $x = 3^t$). This will be 0 unless $x \equiv 5$ mod 7, which is equivalent to $t \equiv 5$ mod 6. We find that two of these polynomials will always yield, between them, the smallest exponent. Similarly to (4.12) and Table 4.13, we write these polynomials as $p_i(3^m + y)$ for carefully-chosen values of m. Much preliminary work is required to discover these values of m. Ignoring unit coefficients and ignoring higher-power terms whose coefficients will be sufficiently divisible that they will not affect the divisibility, these polynomials will be as in Table 5.16.

Table 5.16. *Certain $p_i(3^m + y)$, (linear part only)*

m	p_1	p_2
$5,\ 35$	$7^5 + 7^4 y$	$7^{\geq 12} + 7^{11} y$
$11 + 12 \cdot 7^6$	$7^{12} + 7^4 y$	$7^{12} + 7^{11} y$
$17 + 18 \cdot 7^{12}$	$7^{18} + 7^4 y$	$7^{18} + 7^{11} y$
$23 + 18 \cdot 7^{18}$	$7^{25} + 7^4 y$	$7^{24} + 7^{11} y$
$29 + 12 \cdot 7^{24}$	$7^{31} + 7^4 y$	$7^{30} + 7^{11} y$
$41 + 24 \cdot 7^{36}$	$7^{43} + 7^4 y$	$7^{42} + 7^{11} y$

The claim of the theorem follows from Table 5.16 by the same argument as was used in the proof of 4.10. For t in the specified congruence, if $3^t = 3^m + y$, then $\nu(y) = \nu(t - m) + 1 \geq 2$, similarly to (4.14). For example, if $t \equiv 11$ mod 42, and $3^t = 3^{11+12\cdot 7^6} + y$, then $\nu(y) = \nu(t - 11 - 12 \cdot 7^6) + 1$. Thus $\min(\nu(p_1(3^t)), \nu(p_2(3^t)))$

will be determined by the 7^5 in p_1 if $t \equiv 5, 35$ (42), while in the other cases, it is determined by the $7^4 y$ in p_1 or the constant term in p_2.

The groups $v_1^{-1}\pi_{2t-1}(X_{34})$ are cyclic by an argument similar to the one described at the end of the proof of 4.10, and have the same order as $v_1^{-1}\pi_{2t}(X_{34})$ for the standard reason described there. ■

Similarly to the discussion preceding Conjecture 4.1, one of the factors in the product decomposition of $SU(42)_7$ given in [29] is an H-space $B_5^7(7)$ whose $\mathbb{F}_7$-cohomology is an exterior algebra on classes of grading 11, 23, 35, 47, 59, 71, and 83, and which is built from spheres of these dimensions by fibrations. Using [36], we can obtain a degree-1 map $B_5^7(7) \to S^{71}$. Let $B_7 := B(11, 23, 35, 47, 59, 83)$ denote its fiber.

Conjecture 5.17. *There is an equivalence* $(X_{34})_7 \simeq B_7$.

The evidence for this conjecture is the following analogue of Proposition 4.16.

Proposition 5.18. *There is an isomorphism of Adams modules* $K^*(X_{34}; \hat{\mathbb{Z}}_7) \approx K^*(B_7; \hat{\mathbb{Z}}_7)$.

Proof. We argue exactly as in the proof of 4.16. We compute ψ^3 on the basis $\langle y^5, y^{11}, y^{17}, y^{23}, y^{29}, y^{35}, y^{41}\rangle$ of $QK^1(SU(42))$ where $y = e_p(\ell_0(x))$. Then $(0,0,0,0,0,1,5)^T$ is an eigenvector for $\lambda = 3^{35}$. Quotienting out by this vector, we obtain as the transpose of the matrix of ψ^3 on this basis of $QK^1(B_7; \hat{\mathbb{Z}}_7)$ the following matrix $(\psi^3)^T_{B_7}$. (We write the transpose for typographical reasons.)

$$\begin{pmatrix} 3^5 & -126360 & 118399320 & -136947072600 & 176770713576600 & 354126788968985033040 \\ 0 & 3^{11} & -202656168 & 253117553832 & -340318273704552 & -711333125213838324912 \\ 0 & 0 & 3^{17} & -228319808184 & 356407220575224 & 855037442924642953872 \\ 0 & 0 & 0 & 3^{23} & -225190483754184 & -736156248630810154992 \\ 0 & 0 & 0 & 0 & 3^{29} & 453306387710146810320 \\ 0 & 0 & 0 & 0 & 0 & 3^{41} \end{pmatrix}$$

Let ψ^3_X denote the matrix on X_{34}. Using eigenvectors of the two matrices similarly to the proof of 4.16, we find that the matrix Q below satisfies $\psi^k_X Q = Q\psi^k_{B_7}$ for all k, and has diagonal entries $\not\equiv 0$ mod 7. Finding such a matrix Q was by no means automatic; it required simultaneous satisfying of many congruence equations.

$$Q = \begin{pmatrix} 5/2 & 0 & 0 & 0 & 0 & 0 \\ -95/2 & 66 & 0 & 0 & 0 & 0 \\ -81005 & 138611 & -88219 & 0 & 0 & 0 \\ 232625/2 & -210177 & 173677 & -55946 & 0 & 0 \\ -253775 & 487751 & -507389 & 312395 & -85347 & 0 \\ 301425 & -592328 & 668516 & -528725 & 314505 & 24324 \end{pmatrix}$$

■

References

[1] J. Aguadé, *Constructing modular classifying spaces*, Israel Jour Math **66** (1989) 23-40.

[2] K. K. S. Andersen, *The normalizer splitting conjecture for p-compact groups*, Fund Math **161** (1999) 1-16.

[3] K. K. S. Andersen and J. Grodal, *The classification of 2-compact groups*, preprint, `math.AT/0611437` .

[4] K. K. S. Andersen, J. Grodal, J. M. Moller, and A. Viruel, *The classification of p-compact groups for p odd*, preprint, `math AT/0302346`.

[5] M. Bendersky, *Some calculations in the unstable Adams-Novikov spectral sequence*, **16** (1980) 739-766.

[6] M. Bendersky and D. M. Davis, *v_1-periodic homotopy groups of the Dwyer-Wilkerson space*, preprint, `arXiv.0706.0993`.

[7] ———, *2-primary v_1-periodic homotopy groups of $SU(n)$*, Amer Jour Math **114** (1991) 465-494.

[8] M. Bendersky, D. M. Davis, and M. Mimura, *v_1-periodic homotopy groups of exceptional Lie groups—torsion-free cases*, Trans Amer Math Soc **333** (1992) 115-136.

[9] A. K. Bousfield, *The K-theory localizations and v_1-periodic homotopy groups of H-spaces*, Topology **38** (1999) 1239-1264.

[10] A. K. Bousfield, E. Curtis, D. Kan, D. Quillen, D. Rector, and J. Schlesinger, *The mod p lower central series and the Adams spectral sequence*, Topology **5** (1966) 331-342.

[11] C. Broto and J. M. Moller, *Finite Chevalley versions of p-compact groups*, preprint, `www.math.ku.dk/ moller/preprints/BrM2Draft.pdf`.

[12] N. Castellana, *Quillen Grassmanians as non-modular homotopy fixed points*, Math Zeit **248** (2004) 477-493.

[13] A. Clark and J. Ewing, *The realization of polynomial algebras as cohomology rings*, Pac Jour Math **50** (1974) 425-434.

[14] J. H. Conway and N. J. A. Sloane, *The Coxeter-Todd lattice, the Mitchell group, and related sphere packings*, Math Proc Camb Phil Soc **93** (1983) 421-440.

[15] G. Cooke, J. R. Harper, and A. Zabrodsky, *Torsion-free mod p H-spaces of low rank*, Topology **18** (1979) 349-359.

[16] D. M. Davis, *Representation types and 2-primary homotopy groups of certain compact Lie groups*, Homology, Homotopy, and Applications **5** (2003) 297-324.

[17] ———, *From representation theory to homotopy groups*, Memoirs Amer Math Soc **759** (2002).

[18] ———, *v_1-periodic homotopy groups of $SU(n)$ at odd primes*, Proc London Math Soc **43** (1991) 529-544.

[19] ———, *Computing v_1-periodic homotopy groups of spheres and certain Lie groups*, Handbook of Algebraic Topology, Elsevier (1995) 993-1049.

[20] D. M. Davis and M. Mahowald, *Some remarks of v_1-periodic homotopy groups*, London Math Soc Lecture Notes Series **176** (1992) 55-72.

[21] D. M. Davis and H. Yang, *Tractable formulas for v_1-periodic homotopy groups of $SU(n)$ when $n \le p^2 - p + 1$*, Forum Math **8** (1996) 585-620.

[22] W. G. Dwyer and C. W. Wilkerson, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Annals of Math **139** (1994) 395-442.

[23] H. Hasse, *Die Gruppe der p^n-primären Zahlen für ein Primteiler von p*, Jour Reine Angew Math **176** (1936) 174-183.

[24] A. Jeanneret and A. Osse, *The K-theory of p-compact groups*, Comm Math Helv 72 (1997) 556-581.

[25] R. M. Kane, *The homology of Hopf space*, North-Holland (1988).

[26] G. Kemper et. al., A database of invariant rings, `www.maths.abdn.ac.uk/~bensond/html/archive/kemper-koerding-malle-matzat-vogel-wiese.html`.

[27] R. Levi, *On finite groups and homotopy theory*, Memoirs Amer Math Soc **567** (1995).

[28] H. Maschke, *Ueber die quarternäre, endliche, lineare Substitutionsgruppe der Borchardt'schen Moduln*, Math Annalen **30** (1887) 496-515.

[29] M. Mimura, G. Nishida, and H. Toda, *Mod p decomposition of compact Lie groups*, Publ RIMS **13** (1977) 627-680.

[30] M. Mimura and H. Toda, *Cohomology operations and the homotopy of compact Lie groups—I*, Topology **9** (1970) 317-336.

[31] D. Notbohm, *Spaces with polynomial mod-p cohomology*, Math Proc Camb Phil Soc **126** (1999) 277-292.

[32] K. Potocka, *The number of summands in v_1-periodic homotopy groups of $SU(n)$*, Topology and Applns **153** (2005) 647-677.

[33] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canadian Jour Math **6** (1954) 274-304.

[34] D. P. Sullivan, *Geometric Topology, the 1970 MIT notes*, (2003) Springer-Verlag.

[35] H. Toda, *Composition methods in the homotopy groups of spheres*, Annals of Math Studies, Princeton **49** (1962).

[36] A. Yamaguchi, *On the p-regularity of Stiefel manifolds*, Publ RIMS Kyoto, **25** (1989) 355-380.

Lehigh University, Bethlehem, PA 18015

E-mail address: `dmd1@lehigh.edu`