MANIFOLD PROPERTIES OF PLANAR POLYGON SPACES

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ABSTRACT. We prove that the tangent bundle of a generic space of planar n-gons with specified side lengths, identified under isometry, plus a trivial line bundle is isomorphic to $(n-2)$ times a canonical line bundle. We then discuss consequences for orientability, cobordism class, immersions, and parallelizability.

1. Main results

Let $\ell = (\ell_1, \ldots, \ell_n)$ be an n-tuple of positive real numbers, and let $M(\ell)$ (resp. $\overline{M}(\ell)$) denote the space of n-gons in the plane with successive side lengths $\ell_1, \ldots, \ell_n$, identified under oriented isometry (resp. isometry). These spaces have been studied by many authors. See, for example, [2], [3], [4], [6], [7], [9], [10], [12], or [13]. If there is no subset $S \subset \{1, \ldots, n\}$ such that $\sum_{i \in S} \ell_i = \sum_{i \notin S} \ell_i$, then ℓ is called *generic*, and $M(\ell)$ and $\overline{M}(\ell)$ are $(n-3)$-manifolds. We restrict our attention to generic length vectors.

There is a canonical double cover $p : M(\ell) \to \overline{M}(\ell)$ which identifies a polygon with its reflection across a side. Associated to p is a canonical line bundle ξ over $\overline{M}(\ell)$. Let $\tau(M(\ell))$ and $\tau(\overline{M}(\ell))$ denote tangent bundles of these spaces. Our first theorem is

Theorem 1.1. *There is a vector bundle isomorphism $\tau(\overline{M}(\ell)) \oplus \varepsilon \approx (n-2)\xi$, where ε is a trivial line bundle.*

We were initially led to Theorem 1.1 by an investigation into the Stiefel-Whitney classes of $\tau(\overline{M}(\ell))$. These, of course, follow immediately from that theorem, but the discoveries came in the opposite order. We will give our original proof of the following corollary in Section 4.

Corollary 1.2. *Let $R = w_1(\xi) \in H^1(\overline{M}(\ell); \mathbb{Z}_2)$. The total Stiefel-Whitney class of $\tau(\overline{M}(\ell))$ satisfies*

$$w(\tau(\overline{M}(\ell))) = (1 + R)^{n-2}.$$

We deduce consequences of these results for orientability, cobordism class, immersions in Euclidean space, and parallelizability.

Our first corollary determines orientability of $\overline{M}(\ell)$.

Corollary 1.3. $\overline{M}(\ell)$ *is orientable iff n is even or $\overline{M}(\ell)$ is diffeomorphic to $(S^1)^{n-3}$.*

Proof. Since w_1 is the obstruction to orientability, it is immediate from Corollary 1.2 that $\overline{M}(\ell)$ is orientable iff n is even or $R = 0$. But $R = 0$ iff the double cover $M(\ell) \to \overline{M}(\ell)$ is trivial, and this is true iff $M(\ell)$ is disconnected. It is noted in [7, Rmk 2.8] that $M(\ell)$ is disconnected iff $\overline{M}(\ell)$ is diffeomorphic to an $(n-3)$-torus. ■

Here is another corollary of Theorem 1.2.

Corollary 1.4. *If $R^{n-3} = 0$, then $\overline{M}(\ell)$ is null cobordant. All n-gon spaces $\overline{M}(\ell)$ which have $R^{n-3} \neq 0$ are cobordant to RP^{n-3}.*

Proof. Let $m = n - 3$. The cobordism class of an m-manifold is determined by which m-dimensional Stiefel-Whitney monomials are nonzero. By Corollary 1.2, all m-dimensional Stiefel-Whitney monomials in $H^*(\overline{M}(\ell); \mathbb{Z}_2)$ equal R^m, so if $R^m = 0$, they are all 0. If $R^m \neq 0$, then $w_{i_1}^{a_1} \cdots w_{i_k}^{a_k}$ with distinct i_j's and positive a_j's is nonzero iff all $\binom{m+1}{i_j}$ are odd, which is also true for RP^m. ■

Our third corollary involves immersions in Euclidean space.

Corollary 1.5. *If $2^e + 3 \leqslant n \leqslant 2^{e+1}$ and $R^{2^{e+1}+2-n} \neq 0 \in H^*(\overline{M}(\ell); \mathbb{Z}_2)$, then $\overline{M}(\ell)$ cannot be immersed in $\mathbb{R}^{2^{e+1}-2}$.*

Proof. If such an immersion exists, then the dual Stiefel-Whitney class $\overline{w}_{2^{e+1}+2-n} = 0$, since $\overline{M}(\ell)$ is an $(n-3)$-manifold. By Corollary 1.2, this equals $\binom{-(n-2)}{2^{e+1}+2-n} R^{2^{e+1}+2-n}$. Since $R^{2^{e+1}+2-n}$ is assumed to be nonzero and $\binom{-(n-2)}{2^{e+1}+2-n} \equiv \binom{2^{e+1}-1}{n-3} \neq 0 \in \mathbb{Z}_2$, we obtain a contradiction to the assumed immersion. ■

Note that if $n = 2^e + 3$, this nonimmersion would be optimal, since the 2^e-manifold $\overline{M}(\ell)$ certainly immerses in $\mathbb{R}^{2^{e+1}-1}$.

Perhaps our most interesting result regards the parallelizability of $\overline{M}(\ell)$. The proof of this appears in Section 2.

Theorem 1.6. *Let $\ell = (\ell_1, \ldots, \ell_n)$ be a generic length vector.*

    a. *If n is odd, then $\overline{M}(\ell)$ is parallelizable iff it is diffeomorphic to the $(n-3)$-torus T^{n-3}.*

    b. *If $n = 6$ or 10, $\overline{M}(\ell)$ is parallelizable.*

    c. *Let $n \equiv 0 \pmod 4$ with $n \geqslant 8$. Then $\overline{M}(\ell)$ is parallelizable if it is diffeomorphic to T^{n-3} or the n-dimensional Klein bottle of [5]. If $\ell = (0^{n-5}, 1, 1, 1, 2, 2)$, then the parallelizability of $\overline{M}(\ell)$ is not known. Otherwise, $\overline{M}(\ell)$ is not parallelizable.*

For $n \leqslant 13$, Theorem 1.6 determines the parallelizability of all spaces $\overline{M}(\ell)$ except two, one with $n = 8$ and one with $n = 12$. The 0-lengths in Theorem 1.6(c) are small sides such that if there are k 0's (denoted 0^k), the length of each is less than $1/k$.

We also remark here briefly about the classification of the spaces $\overline{M}(\ell)$ ([10]). These spaces are classified completely, up to diffeomorphism, by their *genetic code*, which is a set of subsets, called *genes*, of $[n] := \{1, \ldots, n\}$. We will define these in Section 2. The genetic codes are listed for $n \leqslant 6$ in [10] and for $n \leqslant 9$ in [11]. For $6 \leqslant n \leqslant 9$, the number of diffeomorphism classes of nonempty n-gon spaces $\overline{M}(\ell)$ is given in Table 1.

TABLE 1. Number of nonempty n-gon spaces $\overline{M}(\ell)$

n	6	7	8	9
	20	134	2469	175427

In Section 3, we discuss how to tell, in terms of the genetic code of $\overline{M}(\ell)$, whether certain powers of R are nonzero. In particular, we determine for each of the 134 7-gon spaces their cobordism class and whether Corollary 1.5 can be used to obtain an optimal nonimmersion of these 4-manifolds in $\mathbb{R}^6$.

2. Proofs

In this section, we prove Theorems 1.1 and 1.6.

En route to proving Theorem 1.1, we will also note the following result, which was pointed out to us by Jean-Claude Hausmann. We thank him for his help on various matters.

Theorem 2.1. *The vector bundle $\tau(M(\ell)) \oplus \varepsilon$ is isomorphic to a trivial bundle.*

Proof of Theorems 2.1 and 1.1. The manifold $M(\ell)$ can be defined as $F^{-1}(\ell_n)$, where $F : (S^1)^{n-1} \to \mathbb{C}$ is defined by

$$F(z_1, \ldots, z_{n-1}) = \sum_{j=1}^{n-1} \ell_j z_j.$$

See, e.g., [6, (1.2)]. Since ℓ is generic, ℓ_n is a regular value of F, i.e., $F^{-1}(\ell_n) \pitchfork \{\pm 1\}^{n-1}$, (see e.g., [8, Thm 3.1]), and, moreover, there is an ϵ-neighborhood U of ℓ_n such that $F^{-1}(U) \pitchfork \{\pm 1\}^{n-1}$. The set $W = F^{-1}(U)$ is an open subset of $(S^1)^{n-1} - \{\pm 1\}^{n-1}$ and is acted on freely by the involution ϕ defined, using complex conjugation, by $\phi(z_1, \ldots, z_{n-1}) = (\overline{z}_1, \ldots, \overline{z}_{n-1})$. The manifold W is parallelizable with vector fields $v_1, \ldots, v_{n-1}$ defined by

$$v_j(z_1, \ldots, z_{n-1}) = (0, \ldots, 0, iz_j, 0, \ldots, 0).$$

Let $\iota : M(\ell) \to W$ denote the inclusion map. Note that the tubular neighborhood W can be considered to be the normal bundle ν of $M(\ell)$, which is trivial by the construction of $M(\ell)$ and W using F. The trivial bundle $\iota^*(\tau(W))$ is isomorphic to $\tau(M(\ell)) \oplus \nu$. We obtain

$$(n-1)\varepsilon \approx \tau(M(\ell)) \oplus 2\varepsilon.$$

This implies Theorem 2.1 by Lemma 2.3.

Let $\overline{W}$ be the quotient of W by the free involution ϕ, and note that $\overline{M}(\ell)$ is the quotient of $M(\ell)$ by ϕ. The double cover p defined above is the restriction to $M(\ell)$ of the double cover $W \to \overline{W}$. Let ξ_W denote the associated line bundle. Then $(n-1)\xi_W$ can be given by $W \times \mathbb{R}^{n-1}/(w, \langle t_j \rangle) \sim (\phi(w), \langle -t_j \rangle)$, and there is a vector bundle isomorphism $\tau(\overline{W}) \approx (n-1)\xi_W$ defined by

$$(w, \sum t_j v_j(w)) \leftrightarrow (w, \langle t_j \rangle).$$

This is well-defined since $\phi_* : \tau(W) \to \tau(W)$ satisfies $\phi_*(v_j(w)) = -v_j(\phi(w))$.

The normal bundle $\bar{\nu}$ of $\overline{M}(\ell)$ is isomorphic to $M(\ell) \times \mathbb{R} \times \mathbb{R}/(x, s, t) \sim (\phi(x), s, -t)$, which is isomorphic to $\varepsilon \oplus \xi$. We obtain

$$(n-1)\xi \approx \bar{\imath}^*(\tau(\overline{W})) \approx \tau(\overline{M}(\ell)) \oplus \bar{\nu} \approx \tau(\overline{M}(\ell)) \oplus \varepsilon \oplus \xi. \qquad (2.2)$$

There exists a vector bundle θ over $\overline{M}(\ell)$ such that $\xi \oplus \theta$ is isomorphic to a trivial bundle. Adding θ to both sides of (2.2), we obtain that $\tau(\overline{M}(\ell)) \oplus \varepsilon$ is stably isomorphic to $(n-2)\xi$. Theorem 1.1 now follows from Lemma 2.3. ■

The following lemma, which is certainly well-known to experts, was used above.

Lemma 2.3. *Let θ and η be stably isomorphic $(m + 1)$-plane bundles over an m-dimensional CW-complex X. Assume also that if $m + 1$ is even, then $w_1(\theta) = 0 = w_1(\eta)$. Then θ and η are isomorphic.*

Proof. Let $B_SO(m + 1)$ (resp. B_SO) be $BSO(m + 1)$ (resp. BSO) if $m + 1$ is even, and $BO(m + 1)$ (resp. BO) if $m + 1$ is odd, and let $G = \mathbb{Z}$ if $m + 1$ is even, and $\mathbb{Z}_2$ if $m + 1$ is odd. Let f and g be the maps $X \to B_SO(m + 1)$ classifying θ and η, and $i : B_SO(m + 1) \to B_SO$ the usual inclusion. The hypothesis is that $i \circ f \simeq i \circ g$. As a fibration, i is orientable in the stable range. (e.g., [15, Cor 5.2(iii)]) Thus i has a Moore-Postnikov tower which, through dimension $m + 1$, is a fiber sequence

$$K(G, m + 1) \to B_SO(m + 1) \to B_SO \to K(G, m + 2).$$

There is an action map $\mu : K(G, m+1) \times B_SO(m+1) \to B_SO(m+1)$, and $g = \mu(c \times f)$ for some $c : X \to K(G, m + 1)$. Since X is m-dimensional, c is trivial, and hence $g \simeq f$. ■

Proof of Theorem 1.6. Part (a) is immediate from the proof of Corollary 1.3, which notes that if n is odd and not diffeomorphic to T^{n-3}, then $w_1(\tau(\overline{M}(\ell))) \neq 0$, so $\overline{M}(\ell)$ is not parallelizable. Part (b) when $n = 6$ follows from Corollary 1.3 and the well-known result ([17]) that every compact orientable 3-manifold is parallelizable.

Now we prove part (b) when $n = 10$. In this case $\overline{M}(\ell)$ and $M(\ell)$ are 7-manifolds, and so the Atiyah-Hirzebruch spectral sequence gives a commutative diagram of short exact sequences

$$0 \longrightarrow H^4(\overline{M}(\ell); \mathbb{Z}) \xrightarrow{\overline{q}} \widetilde{KO}(\overline{M}(\ell)) \longrightarrow F_2(\overline{M}(\ell)) \longrightarrow 0$$

$$\downarrow p_1^* \qquad\qquad \downarrow p_2^* \qquad\qquad \downarrow p_3^*$$

$$0 \longrightarrow H^4(M(\ell); \mathbb{Z}) \xrightarrow{q} \widetilde{KO}(M(\ell)) \longrightarrow F_2(M(\ell)) \longrightarrow 0,$$

in which $F_2(\ )$ is an extension of $H^1(\ ; \mathbb{Z}_2)$ and $H^2(\ ; \mathbb{Z}_2)$, and hence is a group of order 2 or 4.

Since $p : M(\ell) \to \overline{M}(\ell)$ is a double cover, $2\ker(p_1^*) = 0$. The pullback $p^*(\xi)$ is a trivial bundle, and hence $p_2^*([\xi]) = 0$. ⟦This pullback bundle is

$$\{(x, x, t), (x, \phi(x), t) \in M(\ell) \times M(\ell) \times \mathbb{R}\}/(x, x, t) \sim (x, \phi(x), -t),$$

which maps to $\overline{M}(\ell) \times \mathbb{R}$ by sending (x, x, t) and $(x, \phi(x), -t)$ to $([x], t)$.⟧

There exists $\alpha \in H^4(\overline{M}(\ell))$ such that $\overline{q}(\alpha) = [4\xi]$. Since $p_2^*([4\xi]) = 0$, the diagram implies that $\alpha \in \ker(p_1^*)$, and hence $2\alpha = 0$. Thus $[8\xi] = 0$, and hence $\tau(\overline{M}(\ell))$ is stably trivial by Theorem 1.1. Thus $\overline{M}(\ell)$ is stably parallelizable, and hence is parallelizable by [1], which shows that a stably parallelizable 7-manifold is parallelizable.

Part (c) follows from Corollary 1.2 and Theorem 2.4 together with the observation that (a) if $\overline{M}(\ell)$ has genetic code $\{n, n-3, \ldots, 1\}$, then $\overline{M}(\ell)$ is diffeomorphic to T^{n-3}, which is parallelizable, (b) if $\overline{M}(\ell)$ has genetic code $\{n, n-4, \ldots, 1\}$, then it is diffeomorphic to the $(n-3)$-dimensional Klein bottle of [5], which was shown there to be parallelizable if and only if $n-3$ is odd, and (c) if $\ell = (0^{n-5}, 1, 1, 1, 2, 2)$, then the genetic code of $\overline{M}(\ell)$ is $\{n, n-2, n-5, \ldots, 1\}$. ∎

Theorem 2.4. *If $n \geqslant 5$, then $R^2 = 0$ in $H^*(\overline{M}(\ell); \mathbb{Z}_2)$ iff the genetic code of $\overline{M}(\ell)$ is $\{n, n-3, \ldots, 1\}$, $\{n, n-4, \ldots, 1\}$, or $\{n, n-2, n-5, \ldots, 1\}$.*

In order to prove Theorem 2.4, we recall, in Proposition 2.5, our interpretation ([4, Thm 2.1]) of Hausmann-Knudson's determination ([9]) of the algebra $H^*(\overline{M}(\ell); \mathbb{Z}_2)$. All sets in a genetic code contain the integer n, and so we say that a *gee* is a gene with the n omitted, and a *geetic* code is the genetic code without listing n in each of the sets. There is a partial order on the set of genes or gees by $S = \{s_1, \ldots, s_k\} \leqslant T$ if T contains a subset $\{t_1, \ldots, t_k\}$ with $s_i \leqslant t_i$ for all i. A *subgee* is any set of positive integers which is $\leqslant$ some gee, and gees are the maximal subgees.

Proposition 2.5. *$H^*(\overline{M}; \mathbb{Z}_2)$ is generated by 1-dimensional classes R and $V_1, \ldots, V_{n-1}$ with only relations as below, where $V_S := \prod_{i \in S} V_i,$*

- *V_S is zero unless S is a subgee;*
- *$V_i^2 = RV_i$;*
- *If S is a subgee with $|S| \geqslant n - 2 - d$, then*

$$\sum_{T \nearrow S} R^{d-|T|} V_T = 0.$$

We denote the third relation here by $\mathcal{R}_S$.

For the rest of this section, we are dealing with n-gon spaces.

Definition 2.6. *For $G \subseteq [n-1]$, let $\tilde{G} = [n-1] - G$ and $\overline{G} = \tilde{G} - \max\{i : i \in \tilde{G}\}$.*

Lemma 2.7. *If G_1 and G_2 are subsets of $[n-1]$ (possibly equal) with $G_2 \geqslant \overline{G}_1$, then G_1 and G_2 cannot both be subgees of the same genetic code.*

Proof. This is the only place that we need to know the relationship between genes and length vectors. A subset S of $[n]$ is defined to be *short* if $\sum_{i \in S} \ell_i < \sum_{i \notin S} \ell_i$. Then a set $T \subset [n-1]$ is a subgee iff $T \cup \{n\}$ is short. If G_1 is a subgee, then $\tilde{G}_1$ is not short (called *long*). Therefore $\{n\} \cup \overline{G}_1$ is long, and hence so is $\{n\} \cup G_2$. Thus G_2 is not a subgee. ∎

Proof of Theorem 2.4. We introduce some notational shortcuts: $k, , 1$ for $k, \ldots, 1$ and $k, , \hat{i}, , 1$ for $k, \ldots, i+1, i-1, \ldots, 1$. Basically a double comma means $, \ldots, ,$; i.e., include all intermediate numbers. Also, all cohomology groups have coefficients in $\mathbb{Z}_2$.

By Proposition 2.5, the relations in $H^2(\overline{M}(\ell))$ are associated to subgees of size $\geqslant n - 4$. In the next paragraph, we will show that the only possible subgees of size $\geqslant n - 4$ are (a) $\{n-3, , 1\}$; (b) $\{n-4, , 1\}$; (c) $\{n-3, , \hat{i}, , 1\}$ with $\{1 \leqslant i < n - 3\}$; (d) $\{n-2, n-5, , 1\}$; and (e) $\{n-1, n-5, , 1\}$.

We show that $G \geqslant \overline{G}$ in all other cases, and so G is not a subgee by Lemma 2.7. If $G = [n-1] - \{i\}$, then $\overline{G} = \varnothing$, so $G \geqslant \overline{G}$. If $G = [n-1] - \{i, j\}$, $i > j$, then $\overline{G} = \{j\}$, and so $G \geqslant \overline{G}$ unless $(i, j) = (n-1, n-2)$. If $G = [n-1] - \{i, j, k\}$, $i > j > k$, then $\overline{G} = \{j, k\}$, and so $G \geqslant \overline{G}$ unless $(i, j) = (n-1, n-2)$ or $(j, k) = (n-3, n-4)$.

The group $H^2(\overline{M}(\ell))$ is spanned by R^2 and all V_iV_j, where $1 \leqslant i \leqslant j \leqslant k_0$, where k_0 is the largest integer contained in any of the gees of ℓ. If the geetic code of ℓ does not have any gees of length $\geqslant n - 4$, then there are no relations among these classes, and so $R^2 \neq 0$. We now consider geetic codes having a gee of type (a) through (e) above, plus perhaps other gees.

Case (a): If $G = \{n - 3, , 1\}$ appears alone in the geetic code, then $H^2(\overline{M}(\ell))$ is spanned by R^2 and V_iV_j, $1 \leqslant i \leqslant j \leqslant n - 3$. Since none of these sets $\{i, j\}$ is disjoint from G, the relation $\mathcal{R}_G$ is exactly $R^2 = 0$. This G (as G_1) cannot be accompanied in a geetic code by Lemma 2.7, because an accompanying G_2 cannot satisfy $G_2 \geqslant \overline{G} = \{n - 2\}$, but any such G_2 is $\leqslant G_1$, and hence cannot appear separately in the geetic code, since geetic codes only include maximal subgees.

The remaining cases deal with the situation when the largest gee has size $n - 4$. Note that the possible subgees of size $n - 4$ are totally ordered by $\geqslant$, and so the set of subgees of size $n - 4$ will be exactly those which are $\leqslant$ the single gee of size $n - 4$.

Case (b): If $G = \{n - 4, , 1\}$ appears alone in the geetic code, then the relation $R^2 = 0$ is obtained from $\mathcal{R}_G$ as in Case (a). This G can be accompanied in the geetic code by gees G' containing an integer $i > n - 4$, but such G' must have length $< n - 4$, else it would be $> G$, contradicting maximality of G. Thus G' does not add a new relation, but now $\mathcal{R}_G$ says $0 = R^2 + V_i^2$ (plus possibly other V_j^2). Hence $R^2 \neq 0$.

Case (c): If $G = \{n - 3, , \hat{i}, , 1\}$ appears alone in the geetic code, it will have relations $\mathcal{R}_{n-3,,\hat{j},,1}$ for all $j \leqslant i$, which is $R^2 + V_j^2 = 0$, since there are no 2-subsets of $[n - 3]$ disjoint from G. Clearly, no combination of these relations can yield $R^2 = 0$. This G can be accompanied in the geetic code, but, as noted above, not by G' of size $\geqslant n - 4$. Thus there are no additional relations. The accompanying G'''s may add additional basis elements to $H^2(\overline{M}(\ell))$, such as V_{n-2}^2, but these will not affect the impossibility, already noted, of obtaining $R^2 = 0$ as a consequence of the relations.

Case (d): If $G = \{n - 2, n - 5, , 1\}$ appears alone in the geetic code, it will have relations $\mathcal{R}_G$, $\mathcal{R}_{n-3,n-5,,1}$, and $\mathcal{R}_{n-4,,1}$. Then $H^2(\overline{M}(\ell))$ is spanned by classes R^2, $V_1^2, \ldots, V_{n-2}^2$, and V_iV_j with $i > j$ and $j \leqslant n - 5$. No V_iV_j appears in any of the relations. The three relations are $R^2 + V_{n-3}^2 + V_{n-4}^2$, $R^2 + V_{n-2}^2 + V_{n-4}^2$, and $R^2 + V_{n-2}^2 + V_{n-3}^2$. Adding yields $R^2 = 0$. This G can be accompanied in the geetic code, but an accompanying G' cannot be $\geqslant \overline{G} = \{n - 3, n - 4\}$ by the lemma, and

so its second largest element must by $\leqslant n - 5$. It must contain $n - 1$, else it would be $< G$. Then the three relations all contain the term V_{n-1}^2, and so their sum is no longer just R^2.

Case (e): If $G = \{n - 1, n - 5, , 1\}$ appears alone in the geetic code, there are four relations, each of the form $R^2 + T$, where T is the sum of any three of $\{V_{n-1}^2, V_{n-2}^2, V_{n-3}^2, V_{n-4}^2\}$. No combination of these can equal R^2. Any G' which would accompany G in the geetic code cannot be $\geqslant \overline{G} = \{n - 3, n - 4\}$, so its second largest element must be $\leqslant n - 5$, and so it is $\leqslant G$. Thus this G cannot be accompanied in the geetic code. ∎

3. Specific results for 7-gon spaces

The genetic codes of the 134 7-gon spaces $\overline{M}(\ell)$ are listed in [11]. These are connected 4-manifolds, and we can use `Maple` to determine for each whether $R^4 \neq 0$ (which is, by Corollary 1.4, equivalent to it being cobordant to RP^4) and whether $R^3 \neq 0$ (which is equivalent to having Corollary 1.5 imply that it does not immerse in $\mathbb{R}^6$). We first state the results, and then describe the algorithm.

Proposition 3.1. *Of the 134 7-gon spaces, 72 are cobordant to $\varnothing$, and 62 are cobordant to RP^4.*

Proposition 3.2. *Of the 134 7-gon spaces, 122 have $R^3 \neq 0$ and hence cannot be immersed in $\mathbb{R}^6$. The ones with $R^3 = 0$ are those with geetic codes*

21, 41, 61, 65, 321, 421, 521, 621, 4321, $\{321, 51\}$, $\{421, 61\}$, $\{431, 51\}$.

Here we concatenate, and omit $\{-\}$ from monogenic codes; e.g., 421 means $\{4, 2, 1\}$.

By Proposition 2.5, a presentation matrix for $H^{n-3}(\overline{M}(\ell); \mathbb{Z}_2)$ has columns (generators) for all subgees S (including $\varnothing$, which corresponds to R^{n-3}) and rows (relations) for all subgees T except $\varnothing$. An entry is 1 iff S and T are disjoint, else 0. We know that $\dim(H^{n-3}(\overline{M}(\ell); \mathbb{Z}_2)) = 1$, and so, if this presentation matrix is $(r-1)$-by-r, then its rank is $r - 1$. Then $R = 0$ iff a row-reduced form of the matrix has a row with its only 1 being in the R-column, and this is true iff, when the R-column is omitted, the matrix has rank $r - 2$. So we just form the matrix without the R-column and ask `Maple` whether its rank (over $\mathbb{Z}_2$) is less than its number of columns.

Similarly, $H^{n-4}(\overline{M}(\ell); \mathbb{Z}_2)$ has the same columns, but now rows for all subgees of size $\geqslant 2$, filled in according to the same prescription. We know that

$$\dim(H^{n-4}(\overline{M}(\ell); \mathbb{Z}_2)) = \dim(H^1(\overline{M}(\ell); \mathbb{Z}_2)),$$

and this equals the number of subgees of size $\leqslant 1$, and so $R^{n-4} = 0$ iff, when the R-column is removed, the rank of the resulting matrix is one less than its number of rows.

In [2], we determined a formula for R^{n-3} in a monogenic genetic code, for arbitrary n. As noted in Corollary 1.4, the mod-2 value of R^{n-3} determines whether $\overline{M}(\ell)$ is cobordant to $\varnothing$ or to RP^{n-3}.

Proposition 3.3. *If the genetic code of $\overline{M}(\ell)$ is $\{n, g_1, \ldots, g_k\}$, let $a_i = g_i - g_{i+1} > 0$ $(a_k = g_k)$. Then*

$$R^{n-3} = \sum_B \prod_{i=1}^k \binom{a_i+b_i-2}{b_i} \in \mathbb{Z}_2,$$

where $B = (b_1, \ldots, b_k)$ ranges over all k-tuples of nonnegative integers satisfying $b_1 + \cdots + b_\ell \leqslant \ell$ for $1 \leqslant \ell \leqslant k$ with equality if $\ell = k$.

One can tell from the genetic code whether or not $\overline{M}(\ell)$ has a nonzero vector field.

Proposition 3.4. *Let d_i denote the number of subgees of size i. Then an n-gon space $\overline{M}(\ell)$ has a nonzero vector field iff n is even or $\sum_{i \geqslant 0} (-1)^i d_i = 0$.*

Proof. We use the well-known result of Hopf that a connected manifold has a nonzero vector field iff its Euler characteristic is 0. If n is even, the result follows since the Euler characteristic of an odd-dimensional manifold is 0. That the alternating sum of d_i's gives the Euler characteristic of $\overline{M}(\ell)$ appears as a remark at the end of Section 4 of [9]. We prove it by noting that (e.g., [6, Thm 1.7] or [3, Thm 2.3]) the Betti numbers of the double cover $M(\ell)$ are given by counting subgees and their dual classes. Thus, if $\dim(M(\ell))$ is even, $\chi(M(\ell)) = 2\sum(-1)^i d_i$, and $\chi(\overline{M}(\ell)) = \frac{1}{2}\chi(M(\ell))$. ∎

We list the geetic codes of the 30 cases with $n = 7$ that have Euler characteristic 0. This is obtained from [11].

$$1, 21, 31, 41, 51, 61, 321, 421, 521, 621, 431, 4321, \{321, 41\}, \{321, 51\}, \{321, 61\}$$

$$\{421, 51\}, \{421, 61\}, \{431, 51\}, \{431, 61\}, \{521, 61\}, \{32, 4\}, \{42, 6\}, \{32, 41, 5\}, \{32, 51, 6\}$$

$$\{321, 42, 5\}, \{321, 43, 6\}, \{421, 43, 5\}, \{421, 52, 6\}, \{321, 42, 51, 6\}, \{421, 43, 51, 6\}$$

For example, $\{421, 51\}$ has $d_0 = 1$, $d_1 = 5$, $d_2 = 6$ (21, 31, 41, 51, 32, 42), and $d_3 = 2$ (321, 421).

4. ORIGINAL PROOF OF COROLLARY 1.2

As noted in the introduction, we obtained Corollary 1.2 prior to Theorem 1.1. In this section, we give that original proof. Throughout this section, we let $m = n - 3 = \dim(\overline{M}(\ell))$ We use the following well-known relationship between the Stiefel-Whitney classes of the tangent bundle and the Wu classes. (e.g., [14])

Proposition 4.1. *Let M be an m-manifold. The Wu class $v_i \in H^i(M; \mathbb{Z}_2)$ is defined to be the unique class which satisfies $v_i \cup x = \mathrm{Sq}^i(x)$ for all $x \in H^{m-i}(M; \mathbb{Z}_2)$. Then the total Stiefel-Whitney class, $w(\tau(M))$, of the tangent bundle of M equals $\mathrm{Sq}(v)$, where Sq is the total Steenrod square and $v = \sum_{i=0}^{[m/2]} v_i$ is the total Wu class.*

The following key lemma gives a surprisingly simple formula for the Wu classes of $\overline{M}(\ell)$.

Lemma 4.2. $v_i = \binom{m-i}{i} R^i$.

Proof. For this result, all we need to know about $H^*(\overline{M}(\ell); \mathbb{Z}_2)$ is that it is generated as an algebra by 1-dimensional classes $R, V_1, \ldots, V_{n-1}$ with relations $V_a^2 = RV_a \cdot ([9])$ There are additional relations, but we don't need them here. In general, for a product of 1-dimensional classes x_j,

$$\mathrm{Sq}^i(x_1 \cdots x_k) = x_1 \cdots x_k \cdot \sum_{|S|=i} \prod_{j \in S} x_j,$$

where S ranges over all i-subsets of $\{1, \ldots, k\}$. Using the relations $V_a^2 = RV_a$, $H^{m-i}(\overline{M}(\ell); \mathbb{Z}_2)$ is spanned by classes $R^{m-i-j}V_{a_1} \cdots V_{a_j}$, and

$$\mathrm{Sq}^i(R^{m-i-j}V_{a_1} \cdots V_{a_j}) = \binom{m-i}{i} R^{m-j}V_{a_1} \cdots V_{a_j}.$$

These classes may be zero, depending on the other, more complicated relations, but still it is the case that Sq^i acts as multiplication by $\binom{m-i}{i} R^i$.   ∎

Now we can prove Theorem 1.2, using a combinatorial result proved below.

Proof of Corollary 1.2. The first part of (4.3) follows from Proposition 4.1 and Lemma 4.2, while the next-to-last $=$ is Corollary 4.9.

$$w(\tau(\overline{M}(\ell))) = \sum_{j \geqslant 0} \mathrm{Sq}^j \sum_{i \geqslant 0} \binom{m-i}{i} R^i = \sum_{k \leqslant m} R^k \sum_i \binom{m-i}{i}\binom{i}{k-i} = \sum_{k \leqslant m} \binom{m+1}{k} R^k = (1+R)^{m+1},$$

$$(4.3)$$

since $R^{m+1} = 0$.   ∎

In the remainder of this section, we prove the mod-2 combinatorial result, Corollary 4.9, which was used in the above proof. This result and the integral combinatorial results, Lemma 4.4, Corollary 4.7, and Theorem 4.8, which we use to derive it, are probably known, but we could not find them. Nor could we find a proof simpler than the rather elaborate proof that we present here. We use the usual convention that $\binom{m}{k} = m(m-1)\cdots(m-k+1)/k!$ for any integer m and nonnegative integer k.

Lemma 4.4. *If m is an integer, and k a nonnegative integer, then*

$$\sum_{i=0}^{k} \binom{m-i}{i}\binom{i-m+k}{k-i} = \sum_{i=0}^{k+2} \binom{m-i}{i}\binom{i-m+k+2}{k+2-i}.$$

Proof. We use the `Maple` program `Zeil`, as described in [16, ch.6].[1] It discovers that if $f(k,i) = \binom{m-i}{i}\binom{i-m+k}{k-i}$ and

$$G(k,i) = \binom{m-i}{i}\binom{i-m+k}{k-i}\frac{(2k-m+3)(-m+i-1)i(2i-m)}{(k+1-i)(k+2-i)}$$

for $i \leqslant k$, then

$$(k+2)(k-m+1)(f(k,i)-f(k+2,i)) = G(k,i+1)-G(k,i) \quad (4.5)$$

for $i \leqslant k-1$. (We verified this directly in many cases, but for a complete proof, we rely on the software.) Applying $\sum_{i=0}^{k-1}$ to (4.5), we obtain

$$(k+2)(k-m+1)(\Delta-S) = G(k,k)-G(k,0),$$

---

[1]It is called `ct` in [16], but runs as `Zeil` in our implementation.

where Δ is the difference (LHS minus RHS) of the two sums in our lemma, and

$$S = \binom{m-k}{k}\left(1 - \binom{2k-m+2}{2}\right) - \binom{m-k-1}{k+1}\binom{2k-m+3}{1} - \binom{m-k-2}{k+2}.$$

We note that $G(k,0) = 0$. We will show

$$-(k+2)(k-m+1)S = G(k,k), \qquad\qquad (4.6)$$

which implies that $\Delta = 0$, except perhaps if $k - m + 1 = 0$. If $k - m + 1 = 0$, then both sides of the lemma are easily seen to equal 1 if k is odd, and 0 if k is even.

To prove (4.6), we factor out $\binom{m-k}{k}$, and then (4.6) becomes

$$(k+2)(m-k-1)\left(1 - \binom{2k-m+2}{2} + \frac{(m-2k+1)(m-2k)(m-2k-3)}{(m-k)(k+1)}\right.$$
$$\left. - \frac{(m-2k)(m-2k-1)(m-2k-2)(m-2k-3)}{(k+2)(k+1)(m-k)(m-k-1)}\right)$$
$$= (2k-m+3)(-m+k-1)k(2k-m)/2,$$

which was verified symbolically by `Maple`.   ■

Corollary 4.7. *If m is an integer and k a nonnegative integer, then*

$$\sum_{i=0}^{k} \binom{m-i}{i}\binom{i-m+k}{k-i} = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd}. \end{cases}$$

Proof. We easily verify when $k = 0$ and 1, and then apply Lemma 4.4.   ■

Theorem 4.8. *If $d \geqslant k - m$, then*

$$\sum_{i=0}^{k} \binom{m-i}{i}\binom{i+d}{k-i} = \sum_{j=0}^{[k/2]} \binom{m+d-1-2j}{k-2j}.$$

Proof. The proof is by induction on $m + d - k$, and when this is fixed, induction on k. The theorem is valid when $m + d - k = 0$ by Corollary 4.7 since

$$\sum \binom{k-1-2j}{k-2j} = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd}. \end{cases}$$

It is also valid when $k = 0$, since both equal 1. Assume the result for smaller values. Then using Pascal's formula at the beginning and end, and the induction hypothesis

in the middle, we have

$$\sum \binom{m-i}{i}\binom{i+d}{k-i} = \sum \binom{m-i}{i}\left(\binom{i+d-1}{k-i} + \binom{i+d-1}{k-1-i}\right)$$

$$= \sum \left(\binom{m+d-2-2j}{k-2j} + \binom{m+d-2-2j}{k-1-2j}\right) = \sum \binom{m+d-1-2j}{k-2j}.$$

■

Corollary 4.9. *If* $m \geqslant k$, *then*

$$\sum_{i=0}^{k} \binom{m-i}{i}\binom{i}{k-i} \equiv \binom{m+1}{k} \pmod{2}.$$

Proof. By Theorem 4.8, the LHS equals $\sum_j \binom{m-1-2j}{k-2j}$. This equals 1 if $k = 0$ and is $\equiv m + 1$ (2) if $k = m$. Both $\sum_j \binom{m-1-2j}{k-2j}$ and the RHS satisfy Pascal's formula, and they agree when $k = 0$ or m. Hence they are equal.   ■

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