COEFFICIENTS IN POWERS OF THE LOG SERIES

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Abstract. We determine the p-exponent in many of the coefficients of $\ell(x)^t$, where $\ell(x)$ is the power series for $\log(1+x)/x$ and t is any integer. In our proof, we introduce a variant of multinomial coefficients. We also characterize the power series $x/\log(1+x)$ by certain zero coefficients in its powers.

1. Main divisibility theorem

The divisibility by primes of the coefficients in the integer powers $\ell(x)^t$ of the power series for $\log(1+x)/x$, given by

$$\ell(x) := \sum_{i=0}^{\infty}(-1)^i \frac{x^i}{i+1},$$

has been applied in several ways in algebraic topology. See, for example, [1] and [4]. Our main divisibility result, 1.1, says that, in an appropriate range, this divisibility is the same as that of the coefficients of $(1 \pm \frac{x^{p-1}}{p})^t$. Here p is any prime and t is any integer. We denote by $\nu_p(-)$ the exponent of p in an integer, and by $[x^n]f(x)$ the coefficient of x^n in a power series $f(x)$.

Theorem 1.1. *If t is any integer and $m \le p^{\nu_p(t)}$, then*

$$\nu_p\left([x^{(p-1)m}]\ell(x)^t\right) = \nu_p(t) - \nu_p(m) - m.$$

Thus, for example, if $\nu_3(t) = 2$, then, for $m = 1, \ldots, 9$, the exponent of 3 in $[x^{2m}]\ell(x)^t$ is, respectively, 1, 0, -2, -2, -3, -5, -5, -6, and -9, which is the same as in $(1 \pm \frac{x^2}{3})^t$. In Section 3, we will discuss what we can say about $\nu_p([x^n]\ell(x)^t)$ when n is not divisible by $(p-1)$ and $n < (p-1)p^{\nu_p(t)}$.

The motivation for Theorem 1.1 was provided by ongoing thesis work of Karen McCready at Lehigh University, which seeks to apply the result when $p = 2$ to make

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more explicit some nonimmersion results for complex projective spaces described in [4]. Proving Theorem 1.1 led the author to discover an interesting modification of multinomial coefficients.

Definition 1.2. *For an ordered r-tuple of nonnegative integers $(i_1,\dots,i_r)$, we define*

$$c(i_1,\dots,i_r) := \frac{(\sum i_j j)(\sum i_j - 1)!}{i_1!\cdots i_r!}.$$

Note that $c(i_1,\dots,i_r)$ equals $(\sum i_j j)/\sum i_j$ times a multinomial coefficient. Surprisingly, these numbers satisfy the same recursive formula as multinomial coefficients.

Definition 1.3. *For positive integers $k \le r$, let E_k denote the ordered r-tuple whose only nonzero entry is a 1 in position k.*

Proposition 1.4. *If $I = (i_1,\dots,i_r)$ is an ordered r-tuple of nonnegative integers, then*

$$c(I) = \sum_{i_k>0} c(I - E_k). \tag{1.5}$$

If we think of a multinomial coefficient $\binom{\sum i_j}{i_1,\cdots,i_r} := (i_1+\cdots+i_r)!/((i_1)!\cdots(i_r)!)$ as being determined by the unordered r-tuple $(i_1,\dots,i_r)$ of nonnegative integers, then it satisfies the recursive formula analogous to that of (1.5). For a multinomial coefficient, entries which are 0 can be omitted, but that is not the case for $c(i_1,\dots,i_r)$.

Proof of Proposition 1.4. The right hand side of (1.5) equals

$$\begin{aligned}
&\sum_k i_k \frac{(\sum i_j - 2)!}{(i_1)!\cdots(i_r)!}\left(\sum_j i_j j - k\right)\\
&= \frac{(\sum i_j - 2)!}{(i_1)!\cdots(i_r)!}\left(\left(\sum i_k\right)\left(\sum i_j j\right) - \sum i_k k\right)\\
&= \frac{(\sum i_j - 2)!}{(i_1)!\cdots(i_r)!}\left(\sum i_j j\right)\left(\sum i_j - 1\right),
\end{aligned}$$

which equals the left hand side of (1.5). $\square$

Corollary 1.6. *If $\sum i_j > 0$, then $c(i_1,\dots,i_r)$ is a positive integer.*

Proof. Use (1.5) recursively to express $c(i_1,\dots,i_r)$ as a sum of various $c(E_k) = k$. $\square$

Corollary 1.7. *For any ordered r-tuple $(i_1,\dots,i_r)$ of nonnegative integers and any prime p,*

$$\nu_p\left(\sum i_j\right) \le \nu_p\left(\sum i_j j\right) + \nu_p\binom{\sum i_j}{i_1,\cdots,i_r}. \tag{1.8}$$

Proof. Multiply numerator and denominator of the definition of $c(i_1,\dots,i_r)$ by $\sum i_j$ and apply Corollary 1.6. □

The proof of Theorem 1.1 utilizes Corollary 1.7 and also the following lemma.

Lemma 1.9. *If t is any integer and $\sum i_j \le p^{\nu_p(t)}$, then*

$$\nu_p\binom{t}{t-\sum i_j, i_1,\dots,i_r} = \nu_p(t) + \nu_p\binom{\sum i_j}{i_1,\dots,i_r} - \nu_p\left(\sum i_j\right). \tag{1.10}$$

Proof. For any integer t, the multinomial coefficient on the left hand side of (1.10) equals $t(t-1)\cdots(t+1-\sum i_j)/\prod i_j!$, and so the left hand side of (1.10) equals $\nu_p(t(t-1)\cdots(t+1-\sum i_j)) - \sum \nu_p(i_j!)$. Since $\nu_p(t-s)=\nu_p(s)$ provided $0<s<p^{\nu_p(t)}$, this becomes $\nu_p(t)+\nu_p((\sum i_j-1)!) - \sum\nu_p(i_j!)$, and this equals the right hand side of (1.10). □

Proof of Theorem 1.1. By the multinomial theorem,

$$[x^{(p-1)m}]\ell(x)^t = (-1)^{(p-1)m}\sum_I T_I,$$

where

$$T_I = \binom{t}{t-\sum i_j, i_1,\dots,i_r}\frac{1}{\prod(j+1)^{i_j}}, \tag{1.11}$$

with the sum taken over all $I=(i_1,\dots,i_r)$ satisfying $\sum i_j j = (p-1)m$. Using Lemma 1.9, we have

$$\nu_p(T_I) = \nu_p(t) + \nu_p\binom{\sum i_j}{i_1,\dots,i_r} - \nu_p\left(\sum i_j\right) - \sum i_j\nu_p(j+1).$$

If $I = mE_{p-1}$, then $\nu_p(T_I) = \nu_p(t) + 0 - \nu_p(m) - m$. The theorem will follow once we show that all other I with $\sum i_j j = (p-1)m$ satisfy $\nu_p(T_I) > \nu_p(t) - \nu_p(m) - m$. Such I must have $i_j > 0$ for some $j \ne p-1$. This is relevant because $\frac{1}{p-1}j \ge \nu_p(j+1)$ with equality if and only if $j = p-1$. For I such as we are considering, we have

$$\begin{aligned}
& \nu_p(T_I) - (\nu_p(t) - \nu_p(m) - m) && (1.12)\\
= \;& \nu_p\binom{\sum i_j}{i_1, \ldots, i_r} - \nu_p(\sum i_j) - \sum i_j \nu_p(j+1) + \nu_p(\sum i_j j) + \tfrac{1}{p-1}\sum i_j j\\
\geq \;& \sum i_j(\tfrac{1}{p-1}j - \nu_p(j+1))\\
> \;& 0.
\end{aligned}$$

We have used (1.8) in the middle step. □

2. Zero coefficients

While studying coefficients related to Theorem 1.1, we noticed the following result about occurrences of coefficients of powers of the reciprocal log series which equal 0.

Theorem 2.1. *If m is odd and $m > 1$, then $[x^m]\big(\frac{x}{\log(1+x)}\big)^m = 0$, while if m is even and $m > 0$, then $[x^{m+1}]\big(\frac{x}{\log(1+x)}\big)^m = 0$.*

Moreover, this property characterizes the reciprocal log series.

Corollary 2.2. *A power series $f(x) = 1 + \sum\limits_{i\geq 1} c_i x^i$ with $c_1 \neq 0$ has $[x^m](f(x)^m) = 0$ for all odd $m > 1$, and $[x^{m+1}](f(x)^m) = 0$ for all even $m > 0$ if and only if $f(x) = \frac{2c_1 x}{\log(1+2c_1 x)}$.*

Proof. By Theorem 2.1, the reciprocal log series satisfies the stated property. Now assume that f satisfies this property and let n be a positive integer and $\epsilon = 0$ or 1. Since

$$[x^{2n+1}]f(x)^{2n+\epsilon} = (2n+\epsilon)(2n+\epsilon-1)c_1c_{2n} + (2n+\epsilon)c_{2n+1} + P,$$

where P is a polynomial in $c_1, \ldots, c_{2n-1}$, we see that c_{2n} and c_{2n+1} can be determined from the c_i with $i < 2n$. □

Our proof of Theorem 2.1 is an extension of arguments of [1] and [2]. It benefited from ideas of Francis Clarke. It can be derived from results in [3, ch.6], but we have not seen it explicitly stated anywhere.

Proof of Theorem 2.1. Let $m > 1$ and

$$\left(\frac{x}{\log(1+x)}\right)^m = \sum_{i\geq 0} a_i x^i.$$

Letting $x = e^y - 1$, we obtain

$$\left(\frac{e^y - 1}{y}\right)^m = \sum_{i\ge 0} a_i (e^y - 1)^i. \tag{2.3}$$

Let j be a positive integer, and multiply both sides of (2.3) by $y^m e^y/(e^y - 1)^{j+1}$, obtaining

$$\begin{aligned}(e^y - 1)^{m-j-1} e^y &= y^m \sum_{i\ge 0} a_i (e^y - 1)^{i-j-1} e^y \qquad (2.4)\\ &= y^m \left(a_j \frac{e^y}{e^y - 1} + \sum_{i\ne j} \tfrac{a_i}{i-j} \tfrac{d}{dy} (e^y - 1)^{i-j} \right).\end{aligned}$$

Since the derivative of a Laurent series has no y^{-1}-term, we conclude that the coefficient of y^{m-1} on the RHS of (2.4) is $a_j [y^{-1}](1 + \frac{1}{y} \frac{y}{e^y - 1}) = a_j$.

The Bernoulli numbers B_n are defined by $\frac{y}{e^y-1} = \sum \frac{B_n}{n!} y^n$. Since $\frac{y}{e^y-1} + \frac{1}{2} y$ is an even function of y, we have the well-known result that $B_n = 0$ if n is odd and $n > 1$.

Let

$$j = \begin{cases} m & m \text{ odd} \\ m+1 & m \text{ even.} \end{cases}$$

For this j, the LHS of (2.4) equals

$$\begin{cases} 1 + \sum \frac{B_i}{i!} y^{i-1} & m \text{ odd} \\ -\frac{d}{dy}(e^y - 1)^{-1} = -\sum \frac{(i-1)B_i}{i!} y^{i-2} & m \text{ even,} \end{cases}$$

and comparison of coefficient of y^{m-1} in (2.4) implies

$$\begin{cases} a_m = \frac{B_m}{m!} = 0 & m \text{ odd} \\ a_{m+1} = -\frac{mB_{m+1}}{(m+1)!} = 0 & m \text{ even,} \end{cases}$$

yielding the theorem. □

3. Other coefficients

In this section, a sequel to Theorem 1.1, we describe what can be easily said about $\nu_p([x^{(p-1)m+\Delta}]\ell(x)^t)$ when $0 < \Delta < p - 1$ and $m < p^{\nu_p(t)}$. This is not relevant in the motivating case, $p = 2$. Our first result says that these exponents are at least as large as those of $[x^{(p-1)m}]\ell(x)^t$. Here t continues to denote any integer, positive or negative.

Proposition 3.1. *If* $0<\Delta<p-1$ *and* $m<p^{\nu_p(t)}$, *then*
$$\nu_p\left([x^{(p-1)m+\Delta}]\ell(x)^t\right)\ge\nu_p(t)-\nu_p(m)-m.$$

Proof. We consider terms T_I as in (1.11) with $\sum i_j j=(p-1)m+\Delta$. Similarly to (1.12), we obtain
$$\begin{aligned}&\nu_p(T_I)-(\nu_p(t)-\nu_p(m)-m)\\ =\ &\nu_p\binom{\sum i_j}{i_1,\ldots,i_r}-\nu_p\left(\sum i_j\right)-\sum i_j\nu_p(j+1)\\ &+\nu_p(m)+m.\end{aligned}\tag{3.2}$$

For $I=(i_1,\ldots,i_r)$, let
$$\begin{aligned}\widetilde{\nu}_p(I)\ &:=\ \nu_p\binom{\sum i_j}{i_1,\ldots,i_r}-\nu_p\left(\sum i_j\right)\\ &=\ \nu_p\left(\frac{1}{i_j}\binom{\sum i_j-1}{i_1,\ldots,i_j-1,\ldots,i_r}\right),\end{aligned}$$
for any j. Thus
$$\widetilde{\nu}_p(I)\ge-\min_j\nu_p(i_j).\tag{3.3}$$

Ignoring the term $\nu_p(m)$, the expression (3.2) is
$$\ge\widetilde{\nu}_p(I)+\sum i_j(\tfrac{1}{p-1}j-\nu_p(j+1))-\tfrac{\Delta}{p-1}.\tag{3.4}$$

Note that
$$\sum i_j(\tfrac{1}{p-1}j-\nu_p(j+1))-\tfrac{\Delta}{p-1}=m-\sum i_j\nu_p(j+1)$$
is an integer and is greater than -1, and hence is ≥0.

By (3.3), if $\widetilde{\nu}_p(I)=-e$ with $e\ge0$, then all i_j are divisible by p^e. Thus $\sum i_j(\frac{1}{p-1}j-\nu_p(j+1))$ is positive and divisible by p^e. Hence it is $\ge p^e$. Therefore, (3.4) is $\ge-e+p^e-1\ge0$. We obtain the desired conclusion, that, for each I, (3.4), and hence (3.2), is ≥0. $\square$

Finally, we address the question of when does equality occur in Proposition 3.1. We give a three-part result, but by the third it becomes clear that obtaining additional results is probably more trouble than it is worth.

Proposition 3.5. *In Proposition 3.1,*

a. *the inequality is strict* ($\neq$) *if* $m \equiv 0$ (p);

b. *equality holds if* $\Delta = 1$ *and* $m \not\equiv 0, 1$ (p);

c. *if* $\Delta = 2$ *and* $m \not\equiv 0, 2$ (p), *then equality holds if and only if* $3m \not\equiv 5$ (p).

Proof. We begin as in the proof of 3.1, and note that, using (1.8), (3.2) is

$$\geq \nu_p(m) - \tfrac{\Delta}{p-1} + \sum i_j(\tfrac{1}{p-1}j - \nu_p(j+1)) - \nu_p((p-1)m + \Delta). \quad (3.6)$$

(a) If $\nu_p(m) > 0$, then $\nu_p((p-1)m + \Delta) = 0$ and so (3.6) is greater than 0.

In (b) and (c), we exclude consideration of the case where $m \equiv \Delta$ (p) because then $\nu_p((p-1)m + \Delta) > 0$ causes complications.

(b) If $\Delta = 1$ and $m \not\equiv 0, 1$ (p), then for $I = E_1 + mE_{p-1}$, (3.2) equals

$$\nu_p(m+1) - \nu_p(m+1) - m + \nu_p(m) + m = 0,$$

while for other I, (3.6) is

$$0 - \tfrac{1}{p-1} + \sum i_j(\tfrac{1}{p-1}j - \nu_p(j+1)) > 0.$$

(c) Assume $\Delta = 2$ and $m \not\equiv 0, 2$ (p). Then

$$\begin{aligned} & T_{2E_1+mE_{p-1}} + T_{E_2+mE_{p-1}} \\ = \;& \frac{t(t-1)\cdots(t-m-1)}{2!m!}\frac{1}{4p^m} + \frac{t(t-1)\cdots(t-m)}{m!}\frac{1}{3p^m} \\ = \;& (-1)^m \tfrac{t}{p^m}\big(\tfrac{1}{8}(-m-1+A) + \tfrac{1}{3}(1+B)\big) \\ = \;& (-1)^m \tfrac{t}{24p^m}(-3m+5+(3A+8B)). \end{aligned} \quad (3.7)$$

Here A and B are rational numbers which are divisible by p. This is true because $\nu_p(t) > \nu_p(i)$ for all $i \leq m$. Since $p > 3$, (3.7) has p-exponent $\geq \nu_p(t) - m$ with equality if and only if $3m - 5 \not\equiv 0$ (p). Using (3.6), the other terms T_I satisfy

$$\nu_p(T_I) - (\nu_p(t) - m) \geq \sum i_j(\tfrac{1}{p-1}j - \nu_p(j+1)) - \tfrac{2}{p-1} > 0.$$

$\square$

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