Immersions of RP^{2^e-1}

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We prove that RP^{2^e-1} can be immersed in $\mathbb{R}^{2^{e+1}-e-7}$ provided $e \geq 7$. If $e \geq 14$, this is 1 better than previously known immersions. Our method is primarily an induction on geometric dimension, with compatibility of liftings being a central issue.

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This paper is dedicated to Michael Barratt on the occasion of his 81st birthday.

1 Statement of result and background

Our main result is the following immersion theorem for real projective spaces.

Theorem 1.1 *If $e \geq 7$, then RP^{2^e-1} can be immersed in $\mathbb{R}^{2^{e+1}-e-7}$.*

This improves, in these cases, by 1 dimension upon the result of Milgram [8], who proved, by constructing bilinear maps, that if $n \equiv 7 \bmod 8$, then RP^n can be immersed in $\mathbb{R}^{2n-\alpha(n)-4}$, where $\alpha(n)$ denotes the number of 1s in the binary expansion of n. In [2, Theorem 1.2], the first and fourth authors used obstruction theory to prove that if $n \equiv 7 \bmod 8$, then RP^n can be immersed in $\mathbb{R}^{2n-D}$, where $D = 14, 16, 17, 18$ if $\alpha(n) = 7, 8, 9, \geq 10$. That result, with $n = 2^e - 1$, is stronger than ours for $e \leq 12$. If $e \geq 13$, then our result improves on the result of [2] by $e - 13$ dimensions. Thus 1.1 improves on all known results by 1 dimension if $e \geq 14$.

In [6], James proved that RP^{2^e-1} cannot be immersed in $\mathbb{R}^{2^{e+1}-2e-\delta}$ where $\delta = 3, 2, 2, 4$ for $e \equiv 0, 1, 2, 3 \bmod 4$. In [5], Gitler and Mahowald announced an immersion result for RP^{2^e-1} in dimension 1 greater than that of James' nonimmersion, which would have been optimal. However, a mistake in the argument of [5] was pointed out by Crabb and Steer. The approach of our paper was initiated by Mahowald around

1970 in an unpublished attempt to prove an optimal immersion of RP^{2^e-1}. In order to improve our result to this extent, we would need to show compatibility of our liftings with liftings given by the Radon-Hurwitz theorem ([4]).

2 Outline of proof

In this section we outline the proof of 1.1. In subsequent sections, we will fill in details.

If θ is a vector bundle over a compact connected space X, we define the geometric dimension of θ, denoted $\mathrm{gd}(\theta)$, to be the fiber dimension of θ minus the maximum number of linearly independent sections of θ. Equivalently, if $\dim(\theta)=n$, then $\mathrm{gd}(\theta)$ equals the smallest integer k such that the map $X\xrightarrow{\theta}BO(n)$ which classifies θ factors through $BO(k)$. The following lemma is standard (See eg Sanderson [9, Theorem 4.2]). Here and throughout, ξ_n denotes the Hopf line bundle over RP^n. We will often write P^n instead of RP^n, and will denote the stunted space P^n/P^{k-1} as P^n_k.

Lemma 2.1 *Let $\phi(n)$ denote the number of positive integers i satisfying $i\le n$ and $i\equiv 0,1,2,4 \mod 8$. Suppose $n>8$. Then RP^n can be immersed in $\mathbb{R}^{n+k}$ if and only if* $\mathrm{gd}((2^{\phi(n)}-n-1)\xi_n)\le k$.

Thus 1.1 will follow from the following result, to the proof of which the remainder of this paper will be devoted.

Theorem 2.2 *If $e\ge 7$, then* $\mathrm{gd}((2^{2^{e-1}-1}-2^e)\xi_{2^e-1})\le 2^e-e-6$.

The bulk of the work toward proving 2.2 will be a determination of upper bounds for $\mathrm{gd}(2^e\xi_n)$ for all $n\equiv 7 \bmod 8$ by induction on e, starting with $e=7$. A similar method could be employed for all n, but we restrict to $n\equiv 7 \bmod 8$ to simplify the already formidable arithmetic. We let $A_k=RP^{8k+7}$, and denote $\mathrm{gd}(m\xi_{8k+7})$ by $\mathrm{gd}(m,k)$.

The classifying map for $2^e\xi_{8k+7}$ will be viewed as the following composite.

$$(2.3)\qquad A_k\xrightarrow{d}(A_k\times A_k)^{(8k+7)}\hookrightarrow\bigcup_j A_j\times A_{k-j}\xrightarrow{f\times f}BO_{2^{e-1}}\times BO_{2^{e-1}}\to BO_{2^e}.$$

Here d is a cellular map homotopic to the diagonal map, $X^{(n)}$ denotes the n–skeleton of X, and f classifies $2^{e-1}\xi$. We write BO_m for $BO(m)$ for later notational convenience.

As a first step, we would like to use (2.3) to deduce that

$$\mathrm{gd}(2^e,k)\le\max\{\mathrm{gd}(2^{e-1},j)+\mathrm{gd}(2^{e-1},k-j):0\le j\le k\}.$$

In order to make this deduction, we need to know that the liftings of the various $2^{e-1}\xi_{8j+7}$ to various BO_m have been made compatibly.

Definition 2.4 *If θ is a vector bundle over a filtered space $X_0 \subset \cdots \subset X_k$, we say that*

$$\mathrm{gd}(\theta|X_i) \le d_i \text{ compatibly for } i \le k$$

if there is a commutative diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_k \arrow[d] & \\ BO_{d_0} \arrow[r] & BO_{d_1} \arrow[r] & \cdots \arrow[r] & BO_{d_k} \arrow[r] & BO_{\dim(\theta)} \end{tikzcd}$$

where the map $X_k \to BO_{\dim(\theta)}$ classifies θ, and the horizontal maps are the usual inclusions.

Remark 2.5 In our filtered spaces, we always assume that the inclusions are cofibrations.

Remark 2.6 Isomorphism classes of n–dimensional vector bundles over X correspond to homotopy classes of maps of X into BO_n. Thus one would initially say that the diagram in 2.4 commutes up to homotopy. However, by 2.7, we may interpret this diagram, and other homotopy commutative diagrams that occur later, as being strictly commutative. To apply the lemma, we will often, at the outset, replace maps $BO_n \to BO_{n+k}$ by homotopy equivalent fibrations.

Lemma 2.7 *If*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] & B \end{tikzcd}$$

commutes up to homotopy and p is a fibration, then f is homotopic to a map f' such that $p \circ f' = g \circ i$.

Proof Let $H\colon A \times I \to B$ be a homotopy from $p \circ f$ to $g \circ i$. By the definition of fibration, there exists $\widetilde{H}\colon A \times I \to E$ such that $p \circ \widetilde{H} = H$ and $\widetilde{H}|A \times 0 = f$. Then $\widetilde{H}|A \times 1$ is our desired f'. ■

If $X_0 \subset \cdots \subset X_k$ and $Y_0 \subset \cdots \subset Y_k$ are filtered spaces, we define, for $0 \le i \le k$,

$$(X \times Y)_i := \bigcup_{j=0}^{i} X_j \times Y_{i-j}.$$

Then $(X \times Y)_0 \subset \cdots \subset (X \times Y)_k$ is clearly a filtered space. We will prove the following general result in Section 3.

Proposition 2.8 *Suppose* $\mathrm{gd}(\theta|X_i) \leq d_i$ *compatibly for* $i \leq k$ *and* $\mathrm{gd}(\eta|Y_i) \leq d'_i$ *compatibly for* $i \leq k$*. For* $0 \leq j \leq k$*, let* $e_j = \max(d_i + d'_{j-i} : 0 \leq i \leq j)$*. Then* $\mathrm{gd}(\theta \times \eta|(X \times Y)_j) \leq e_j$ *compatibly for* $j \leq k$*. Moreover, if* $X = Y$ *and* $\theta = \eta$*, then the maps* $(X \times X)_j \xrightarrow{f} BO_{e_j}$ *can be chosen to satisfy* $f \circ T = f$*, where* $T\colon\ X \times X \to X \times X$ *interchanges factors.*

We will begin an induction by deriving in 4.1 some compatible bounds for $\mathrm{gd}(128, i)$. Proposition 2.8 will, after restriction under the diagonal map, allow us to prove $\mathrm{gd}((\sum 2^{e_i})\xi_n) \leq \max\{\sum \mathrm{gd}(2^{e_i}\xi_{m_i}) : \sum m_i = n\}$. These bounds are not yet strong enough to yield new immersion results. We must improve the bounds by taking advantage of paired obstructions. The following result will be proved in Section 3.

Proposition 2.9 *Let* $BO_n[\rho]$ *denote the pullback of* BO_n *and the* $(\rho-1)$*-connected cover* $BO[\rho]$ *over* BO*, and let* $s = \min(\rho + 2m - 1, 4m - 1)$*.*

(1) *There are equivalences* c'_1 *and* c'_2 *such that the following diagram commutes.*

$$\begin{tikzcd} BO_{2m}[\rho]^{(s)} \arrow[r, "q_1"] \arrow[d, "p_2"'] & (BO_{2m}[\rho]/BO_{2m-1}[\rho])^{(s)} \arrow[r, "c'_1"] \arrow[d, "p'_2"'] & S^{2m} \arrow[d, "i"'] \\ BO_{2m+1}[\rho]^{(s)} \arrow[r, "q_2"] & (BO_{2m+1}[\rho]/BO_{2m-1}[\rho])^{(s)} \arrow[r, "c'_2"] & \Sigma P^{2m}_{2m-1}. \end{tikzcd}$$

Preparatory to the next two parts, we expand this diagram as follows, with $c_i = c'_i \circ q_i$ *and* (X, A) *a finite CW pair.*

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "j"'] & BO_{2m-1}[\rho]^{(s)} \arrow[d, "p_1"'] & \\ X & BO_{2m}[\rho]^{(s)} \arrow[r, "c_1"] \arrow[d, "p_2"'] & S^{2m} \arrow[d, "i"'] \\ & BO_{2m+1}[\rho]^{(s)} \arrow[r, "c_2"] & \Sigma P^{2m}_{2m-1}. \end{tikzcd}$$

(2) *Suppose* $\dim(X) < s$*, and we are given* $X \xrightarrow{f} BO_{2m}[\rho]^{(s)}$ *such that* $f \circ j = p_1 \circ f_1$ *and* $c_1 \circ f$ *factors as* $X \to X/A \xrightarrow{g} S^{2m}$ *with* $[g]$ *divisible by 2 in* $[X/A, S^{2m}]$*.*[1] *Then* $p_2 \circ f$ *lifts to a map* $X \xrightarrow{\ell} BO_{2m-1}[\rho]^{(s)}$ *whose restriction to* A *equals* f_1*.*

(3) *Suppose, on the other hand,* $\dim(X) \leq s$*, and we are given* $X \xrightarrow{f'} BO_{2m+1}[\rho]^{(s)}$ *such that* $f' \circ j = p_2 \circ p_1 \circ f_1$ *and* $c_2 \circ f'$ *factors as* $X \to X/A \xrightarrow{g'} \Sigma P^{2m}_{2m-1}$ *with* $[\Sigma g']$ *divisible by 2 in the stable group* $[\Sigma X/A, \Sigma^2 P^{2m}_{2m-1}]$*. Then* f' *is homotopic rel* A *to a map which lifts to* $BO_{2m}[\rho]^{(s)}$*.*

[1] Note that $[X/A, S^{2m}]$ is in the stable range, from which it gets its group structure.

In Section 4, we will implement 2.8 and 2.9 to prove that the last part of the following important result follows by induction on e from the first five parts and its validity when $e = 7$, while in Section 5, we will establish the first five parts.

Theorem 2.10 *There is a function* $g(e,k)$ *defined for* $e \geq 7$ *and* $k \geq 0$ *satisfying the following.*

(1) *If* $k \geq 2^{e-3}$*, then* $g(e,k) = 2^e$*.*

(2) *For all* e*,* $g(e,0) = g(e,1) = 0$*, and, if* $2 \leq k \leq 2^e$*, then* $g(e,k) \geq 4k+4$*.*

(3) *If* $0 \leq \ell \leq k/2$*, then* $g(e+1,k) \geq g(e,\ell) + g(e,k-\ell) - 1$*.*

(4) *If, for some* ℓ *with* $0 \leq \ell \leq k/2$*, we have* $g(e+1,k) = g(e,\ell)+g(e,k-\ell)-1$*, then, for all* ℓ *with* $0 \leq \ell \leq (k-1)/2$*, we have* $g(e,\ell)+g(e,k-1-\ell) < g(e+1,k)$ *and, if also* k *is even, then* $g(e+1,k) \geq 2g(e,k/2)+1$*.*

(5) *For all* e *and* k*,* $g(e,k) \geq g(e,k-1)$*.*

(6) $\mathrm{gd}(2^e,k) \leq g(e,k)$ *compatibly for all* k*.*

The function g will be defined in (5.1) and 5.5. In Table 1, we list its values for small values of the parameters. We prefer not to tabulate the values $g(e,k) = 2^e$ when $k > 2^{e-3}$.

In Section 6, we apply the basic induction argument, 2.8, and the results for $\mathrm{gd}(2^e\xi)$ in 2.10 to prove the following result by induction on t. This clearly implies 2.2 and hence 1.1.

Proposition 2.11 *For* $e \geq 7$ *and* $t \geq 0$*,* $\mathrm{gd}((2^e+2^{e+1}+\cdots+2^{e+t})\xi_{2^e-1}) \leq 2^e-e-6$*.*

3 Proof of general lifting results

In this section, we prove 2.8 and 2.9. For the first one, we find it more convenient to work with sections rather than geometric dimension.

Theorem 3.1 *Let* $X_0 \subset \cdots \subset X_k$ *and* $Y_0 \subset \cdots \subset Y_k$ *be filtered spaces, and let* θ *(resp.* η*) be a vector bundle over* X_k *(resp.* Y_k*). Suppose given* m_0 *(resp.* n_0*) sections of* θ *on* X_k *(resp.* η *on* Y_k*), of which the first* m_i *(resp.* n_i*) are linearly independent (l.i.) on* X_i *(resp.* Y_i*) for* $0 \leq i \leq k$*. Let*

$$p_j = \min(m_i + n_{j-i} : 0 \leq i \leq j).$$

	k															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
7	0	16	19	32	35	48	51	64	67	80	83	96	99	112	115	128
8	0	15	18	32	34	47	50	64	66	79	82	96	98	111	114	128
e 9	0	14	17	31	33	46	49	64	66	78	81	95	97	110	113	128
10	0	13	16	30	32	45	48	63	65	77	80	94	96	109	112	128
11	0	12	16	29	31	44	47	62	64	76	79	93	95	108	111	127
12	0	12	16	28	30	43	46	61	63	75	78	92	94	107	110	126
13	0	12	16	27	29	42	45	60	62	74	77	91	93	106	109	125
14	0	12	16	26	28	41	44	59	61	73	76	90	92	105	108	124
	k															
	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
8	130	143	146	160	162	175	178	192	194	207	210	224	226	239	242	256
9	130	142	145	159	161	174	177	192	194	206	209	223	225	238	241	256
e 10	130	141	144	158	160	173	176	191	193	205	208	222	224	237	240	256
11	129	140	143	157	159	172	175	190	192	204	207	221	223	236	239	256
12	128	139	142	156	158	171	174	189	191	203	206	220	222	235	238	255
13	127	138	141	155	157	170	173	188	190	202	205	219	221	234	237	254
14	126	137	140	154	156	169	172	187	189	201	204	218	220	233	236	253

Table 1: Values of $g(e,k)$ when $e \le 14$ and $k \le 32$.

Let

$$W_j = \bigcup_{i=0}^{j} X_i \times Y_{j-i}.$$

Then there are p_0 sections of $\theta \times \eta$ on W_k of which the first p_j are linearly independent on W_j for $0 \le j \le k$. Moreover, if $\ell + i \ge j$ and $m_\ell + n_i \ge p_j$, then the first p_j sections are l.i. on $X_\ell \times Y_i$.

Note that we have $m_0 \ge \cdots \ge m_k$, $n_0 \ge \cdots \ge n_k$, and $p_0 \ge \cdots \ge p_k$.

The following result will be used in the final step of the proof of 3.1.

Lemma 3.2 *Suppose θ is an n–dimensional trivial vector bundle over a space X with l.i. sections $t_1, \ldots, t_n$. Suppose $s_1, \ldots, s_r$ are l.i. sections of θ, each of which is a linear combination with constant coefficients of the t_i. Then there is a set $s_1, \ldots, s_r, s'_{r+1}, \ldots, s'_n$ of linearly independent sections of θ, with all these sections being linear combinations with constant coefficients of the t_i.*

Proof Because of the constant-coefficient assumption, this is just a consequence of the result for vector spaces, that a basis for a subspace can be extended to a basis for the whole space. ■

Note that the assumption about constant coefficients was required. For example, the section $s(x) = (x, x)$ of $S^2 \times \mathbb{R}^3$ cannot be extended to a set of three l.i. sections.

Proof of 3.1 Let $r_1, \ldots, r_{m_0}$ be the given sections of θ on X_k, and $s_1, \ldots, s_{n_0}$ the given sections of η on Y_k. These are considered as sections of $\theta \times \eta$ by using 0 on the other component. Clearly $\{r_1, \ldots, r_{m_0}, s_1, \ldots, s_{n_0}\}$ is a set of p_0 sections on W_k which is linearly independent on W_0. The proof will proceed by finding p_1 linear combinations, always with constant coefficients, of these sections which are l.i. on W_1, then p_2 linear combinations of these new sections which are l.i. on W_2, etc, until going into the last stage we have p_{k-1} sections which are l.i. on W_{k-1}, and we find p_k linear combinations of them which are l.i. on W_k. Now we apply the lemma repeatedly, starting with the last p_k sections. At the first step, we extend this set to a set of p_{k-1} sections l.i. on W_{k-1}, and continue until going into the last stage, where we have p_1 sections which are combinations of the original p_0 sections and satisfy the conclusion of the theorem for $1 \le i \le k$. We apply the lemma one last time to extend the set of p_1 sections to the desired set of p_0 sections.

Here is an explicit algorithm for the sections described in the first half of the preceding paragraph. We may assume without loss of generality that $m_0 \ge n_0$.

For j from 0 to k,

- For i from 1 to $p_j - n_0$ (resp. $p_j - m_0$), let $r_i^{(j)} = r_i$ (resp. $s_i^{(j)} = s_i$). (Note that if $n_0 \ge p_j$, then nothing happens at this step.)
- For i from $\max(1, p_j - n_0 + 1)$ to $\min(m_0, p_j)$, let both $r_i^{(j)}$ and $s_{p_j+1-i}^{(j)}$ equal $r_i^{(j-1)} + s_{p_j+1-i}^{(j-1)}$.
- Then the sections $r_i^{(j)}$ and $s_i^{(j)}$ constructed in the two previous steps give the sections which are l.i. on W_j. (Each section constructed in the second step can be counted as an r or an s, but is only counted once.)

We must show that these have the required linear independence. Before doing so, we illustrate with an example, computed by `Maple`. Let $k = 4$, $[m_0, \ldots, m_4] = [11, 6, 4, 1, 0]$ and $[n_0, \ldots, n_4] = [10, 8, 3, 2, 0]$. Then $[p_0, \ldots, p_4] = [21, 16, 14, 9, 7]$. The 16 sections l.i. on W_1 are

$$r_1, \ldots, r_6,\ r_7 + s_{10},\ r_8 + s_9,\ r_9 + s_8,\ r_{10} + s_7,\ r_{11} + s_6,\ s_5, \ldots, s_1.$$

The 14 sections l.i. on W_2 are

$$r_1,\ r_2,\ r_3,\ r_4,\ r_5+r_7+s_{10},\ r_6+r_8+s_9,\ r_7+r_9+s_{10}+s_8,\ r_8+r_{10}+s_9+s_7,$$
$$r_9+r_{11}+s_8+s_6,\ r_{10}+s_7+s_5,\ r_{11}+s_6+s_4,\ s_3,\ s_2,\ s_1.$$

The 9 sections l.i. on W_3 are

$$r_1+r_6+r_8+s_9,\ r_2+r_7+r_9+s_{10}+s_8,\ r_3+r_8+r_{10}+s_9+s_7,$$
$$r_4+r_9+r_{11}+s_8+s_6,\ r_5+r_7+r_{10}+s_{10}+s_7+s_5,$$
$$r_6+r_8+r_{11}+s_9+s_6+s_4, r_7+r_9+s_{10}+s_8+s_3,\ r_8+r_{10}+s_9+s_7+s_2,$$
$$r_9+r_{11}+s_8+s_6+s_1.$$

The 7 sections l.i. on W_4 are

$$\begin{aligned}
&r_1+r_3+r_6+2r_8+r_{10}+2s_9+s_7,\\
&r_2+r_4+r_7+2r_9+r_{11}+s_{10}+2s_8+s_6,\\
&r_3+r_5+r_7+r_8+2r_{10}+s_{10}+s_9+2s_7+s_5,\\
&r_4+r_6+r_8+r_9+2r_{11}+s_9+s_8+2s_6+s_4,\\
&r_5+2r_7+r_9+r_{10}+2s_{10}+s_8+s_7+s_5+s_3,\\
&r_6+2r_8+r_{10}+r_{11}+2s_9+s_7+s_6+s_4+s_2,\\
&r_7+2r_9+r_{11}+s_{10}+2s_8+s_6+s_3+s_1.
\end{aligned}$$

Now we continue with the proof. The property described in the first paragraph of the proof, that the sections claimed to be l.i. on W_j are linear combinations with constant coefficients of those on W_{j-1}, is clear from their inductive definition.

Next we easily show that if $i > p_j - n_0$, then

$$r_i^{(j)} = s_{p_j+1-i}^{(j)} = r_i + \sum_{\ell>i} c_\ell r_\ell + s_{p_j+1-i} + \sum_{\ell>p_j+1-i} d_\ell s_\ell$$

with c_ℓ and d_ℓ integers. The point here is that the additional terms have subscript greater than i or p_j+1-i. The proof is immediate from the inductive formula

$$r_i^{(j)} = r_i^{(j-1)} + s_{p_j+1-i}^{(j-1)}$$

and the fact that $p_j \le p_{j-1}$. Indeed, from $r_i^{(j-1)}$ we obtain terms $r_{\ge i}$ and $s_{\ge p_{j-1}+1-i}$, and from $s_{p_j+1-i}^{(j-1)}$ we obtain terms $s_{\ge p_j+1-i}$ and $r_{\ge p_{j-1}-p_j+i}$.

Finally we show that the asserted sections are l.i. on W_j. Let $\mathbf{x} \in X_\ell \times Y_{j-\ell}$. Note that $\{r_1(\mathbf{x}), \dots, r_{m_\ell}(\mathbf{x})\}$ is l.i., as is $\{s_1(\mathbf{x}), \dots, s_{n_{j-\ell}}(\mathbf{x})\}$, and that $p_j \le m_\ell + n_{j-\ell}$. If we form a matrix with columns labeled

$$r_1, \dots, r_{m_0}, s_{n_0}, \dots, s_1,$$

and rows which express the sections, ordered as

$$r_1^{(j)}, \dots, r_{\min(m_0,p_j)}^{(j)}, s_{p_j-m_0}^{(j)}, \dots, s_1^{(j)}, \tag{3.3}$$

in terms of the column labels, then, by the previous paragraph, the number of columns is $\geq$ (usually strictly greater than) the number of rows, the entry in position (i,i) is 1 for $i \leq \min(m_0, p_j)$, and all entries to the left of these 1s are zero. If $i > \min(m_0, p_j)$, then all entries in the r-portion of row i are zero. Moreover an analogous statement is true if the order of the rows and of the columns are both reversed. Thus there are 1s on the diagonal running up from the lower right corner of the original matrix (for $\min(n_0, p_j)$ positions) and zeros to their right.

If a linear combination of our sections applied to $\mathbf{x}$ is 0, then the triangular form of the matrix implies that the first m_ℓ coefficients are 0, while the triangular form looking up from the lower right corner implies that the last $n_{j-\ell}$ coefficients are 0. Since $p_j \leq m_\ell + n_{j-\ell}$, this implies that all coefficients are 0, hence the desired independence.

The same argument works for the last statement of the proposition. For k satisfying $j \leq k \leq \ell + i$, replace W_k by $W_k \cup (X_\ell \times Y_i)$. Then everything goes through as above. ■

Proof of 2.8 Let $D = \dim(\theta)$ and $D' = \dim(\eta)$. Then d_i, d_i', e_i, and $(X \times Y)_i$ of 2.8 correspond to $D - m_i$, $D' - n_i$, $D + D' - p_i$, and W_i of 3.1, respectively. The compatible gd bounds may be interpreted as vector bundles θ_i over X_i of dimension d_i and isomorphisms $\theta|X_i \approx \theta_i \oplus (D - d_i)$ and $\theta_i|X_{i-1} \approx \theta_{i-1} \oplus (d_i - d_{i-1})$. The trivial subbundles yield, for all i, $D - d_i$ l.i. sections of θ on X_i such that the restrictions of the sections on X_i to X_{i-1} are a subset of the sections on X_{i-1}. Each of the sections on X_0 has a largest X_i for which it is one of the given l.i. sections. By Atiyah [1, Section 1.4.1], this section on X_i can be extended over X_k (although probably not as part of a linearly independent set). Analogous statements are true for sections of $\eta|Y_i$.

By 3.1, there are $D + D' - e_0$ l.i. sections of $\theta \times \eta$ on W_0 of which the first $D + D' - e_i$ are l.i. on W_i. Taking orthogonal complements of the spans of the sections yields the desired compatible bundles on W_i of dimension e_i, yielding the first part of 2.8.

For the second part, first note that in the algorithm in the proof of 3.1, if the r's and s's are equal, then the set of sections constructed on each W_i is invariant under the interchange map T. Thus the same will be true of the orthogonal complement of their span. ■

Proof of 2.9

(1) Let $F_1 = S^{2m-1}$ denote the fiber of $BO_{2m-1}[\rho] \to BO_{2m}[\rho]$. There is a relative Serre spectral sequence for

$$(CF_1, F_1) \to (BO_{2m}[\rho], BO_{2m-1}[\rho]) \to BO_{2m}[\rho]. \tag{3.4}$$

The fibration $V_{2m} \to BO_{2m}[\rho] \to BO[\rho]$ shows that the bottom class of $BO_{2m}[\rho]$ is in dimension $\geq \min(\rho, 2m)$. The spectral sequence of (3.4) shows that $H_*(S^{2m}) \to H_*(BO_{2m}[\rho]/BO_{2m-1}[\rho])$ has cokernel beginning in dimension $\geq s+1$, and so the map is an s–equivalence. Thus the inclusion of the s–skeleton of $BO_{2m}[\rho]/BO_{2m-1}[\rho]$ factors through S^{2m} to yield the map c_1', which is an equivalence.

The second map is obtained similarly. A map

$$\Sigma P^{2m}_{2m-1} \xrightarrow{\ell} BO_{2m+1}[\rho]/BO_{2m-1}[\rho]$$

is obtained as the inclusion of a skeleton of CF_2/F_2, where $F_2 = V_{2m+1,2}$ is the fiber of $BO_{2m-1}[\rho] \to BO_{2m+1}[\rho]$. The relative Serre spectral sequence of

$$(CF_2, F_2) \to (BO_{2m+1}[\rho], BO_{2m-1}[\rho]) \to BO_{2m+1}[\rho] \tag{3.5}$$

implies that $\operatorname{coker}(\ell_*)$ begins in dimension $\geq s+1$, determined by

$$H_{2m}(CF_2, F_2) \otimes H_{\min(\rho, 2m+1)}(BO_{2m+1}[\rho])$$

and the first "product" class in $H_{4m}(\Sigma V_{2m+1,2})$. The obtaining of c_2' now follows exactly as for c_1'.

(2) Let $Q := BO_{2m+1}[\rho]/BO_{2m-1}[\rho]$ and $E := \text{fiber}(BO_{2m+1}[\rho] \to Q)$. The commutative diagram of fibrations

$$\begin{tikzcd}
V_{2m+1,2} \arrow[r] \arrow[d] & BO_{2m-1}[\rho] \arrow[r] \arrow[d] & BO_{2m+1}[\rho] \arrow[d] \\
\Omega Q \arrow[r] & E \arrow[r] & BO_{2m+1}[\rho]
\end{tikzcd}$$

implies the quotient $E/BO_{2m-1}[\rho]$ has the same connectivity as $\Omega Q/V_{2m+1,2}$, which is 1 less than that determined from (3.5); that is, $E/BO_{2m-1}[\rho]$ is $(s-1)$–connected. Thus, since $\dim(X) < s$, the vertical maps in

$$\begin{tikzcd}
BO_{2m-1}[\rho]^{(s)} \arrow[r] \arrow[d] & BO_{2m+1}[\rho]^{(s)} \arrow[r] \arrow[d] & \Sigma P^{2m}_{2m-1} \arrow[d] \\
E \arrow[r] & BO_{2m+1}[\rho] \arrow[r] & Q
\end{tikzcd}$$

are equivalences in the range relevant for maps from X, A, and X/A. Since the bottom row is a fibration, we may consider the top row to be one, too, as far as X is concerned.

Since g is divisible by 2, and $2\pi_{2m}(\Sigma P_{2m-1}^{2m}) = 0$, we deduce that the composite

$$X/A \xrightarrow{g} S^{2m} \xrightarrow{i} \Sigma P_{2m-1}^{2m}$$

represents the 0 element of $[X/A, \Sigma P_{2m-1}^{2m}]$; ie the map is null-homotopic rel $*$. There is a commutative diagram as in (3.6) with the left sequence a cofiber sequence and the right sequence a fiber sequence in the range of $\dim(X)$.

$$\begin{array}{ccc} A & \xrightarrow{f_1} & BO_{2m-1}[\rho]^{(s)} \\ {\scriptstyle j_1}\downarrow & & {\scriptstyle j_2}\downarrow \\ X & \xrightarrow{p_2\circ f} & BO_{2m+1}[\rho]^{(s)} \\ {\scriptstyle q}\downarrow & & \downarrow \\ X/A & \xrightarrow{i\circ g} & \Sigma P_{2m-1}^{2m} \end{array} \tag{3.6}$$

We have just seen that there is a basepoint-preserving homotopy

$$H\colon\ X/A \times I \to \Sigma P_{2m-1}^{2m}$$

from $i \circ g$ to a constant map. There is a commutative diagram

$$\begin{array}{ccccc} X \times 0 \cup A \times I & & \longrightarrow & & BO_{2m+1}[\rho] \\ \downarrow & & & & \downarrow \\ X \times I & \xrightarrow{q\times I} & X/A \times I & \xrightarrow{H} & \Sigma P_{2m-1}^{2m} \end{array}$$

where the top map is $p_2 \circ f$ on $X \times 0$ and $j_2 \circ f_1$ on each $A \times \{t\}$. By the Relative Homotopy Lifting Property of a fibration, there exists a map $\widetilde{H}\colon\ X \times I \to BO_{2m+1}[\rho]$ making both triangles commute. When $t = 1$, it maps into $BO_{2m-1}[\rho]$, since it projects to the constant map at the basepoint of ΣP_{2m-1}^{2m}.

(3) We use the fact that $2 \cdot 1_{\Sigma P_{2m-1}^{2m}}$ factors as

$$\Sigma P_{2m-1}^{2m} \xrightarrow{\text{col}} S^{2m+1} \xrightarrow{\eta} S^{2m} \hookrightarrow \Sigma P_{2m-1}^{2m}$$

to deduce that the composite

$$\Sigma X/A \xrightarrow{\Sigma g'} \Sigma^2 P_{2m-1}^{2m} \xrightarrow{\text{col}} S^{2m+2}$$

is null-homotopic since $[\Sigma g']$ is divisible by 2. Note that we needed to suspend once since if $\dim(X) = 4m-1$, then $[X/A, \Sigma P_{2m-1}^{2m}]$ might not have a group structure. Since

$$[X/A, S^{2m+1}] \xrightarrow{\Sigma} [\Sigma(X/A), S^{2m+2}]$$

is bijective, we deduce that $X/A \xrightarrow{\mathrm{col}\circ g'} S^{2m+1}$ is null-homotopic.

An argument similar to the one in the beginning of the proof of (2) shows that $BO_{2m}[\rho] \to BO_{2m+1}[\rho] \to S^{2m+1}$ is a fibration through dimension $\min(\rho + 2m-1, 4m) \geq s$. Since $\dim(X) \leq s$, the lifting follows as in the proof of (2). ■

4 Inductive determination of a bound for $\mathrm{gd}(2^e, k)$

In this section, we prove that part (6) of 2.10 follows from its first five parts, together with its validity for $e = 7$. We begin by proving the validity when $e = 7$. The following result is stronger than the required liftings for $e = 7$; i.e., we have $m(k) \leq g(7, k)$ and the inequality is strict if k is even with $4 \leq k \leq 14$. The reason for beginning our induction with liftings weaker than the best results that we are able to prove is to fit them into a simple formula that works for all values of e. Here and throughout we use the standard notation that $\nu(-)$ denotes the exponent of 2 in an integer.

Theorem 4.1 *Let*

$$m(k) = \begin{cases} 0 & k = 0, 1 \\ 16 & k = 2 \\ 8k - 5 & k \text{ odd, } 3 \leq k \leq 15 \\ 8k + \nu(k) - 4 & k \text{ even, } 4 \leq k \leq 16. \end{cases}$$

There are compatible liftings of $128\xi_{8k+7}$ *to* $BO_{m(k)}$ *for* $k \geq 0$.

Proof Let H_k denote the Hopf bundle over quaternionic projective space HP^k. Let $m'(k) = 13$ if $k = 2$, and otherwise $m'(k) = m(k)$. We will use [3, Theorem 1.1b] to prove

(4.2) there are compatible liftings of $32H_{2k+1}$ to $BO_{m'(k)}$ for $2 \leq k \leq 16$.

Three things are required to prove this. First we need that, for $k \leq 15$ and all $i \leq 2k+1$ satisfying also $4i - 1 \geq m'(k)$,

$$\nu\tbinom{32}{i} \geq \nu(|\pi_{4i-1}(P_{m'(k)} \wedge bo)|).$$

This is easily verified using $\nu\binom{32}{i} = 5 - \nu(i)$ and, for $1 \leq \epsilon \leq 3$,

$$\nu(|\pi_{4i-1}(P_{4a+\epsilon} \wedge bo)|) = \begin{cases} 4 - \epsilon & i = a + 1 \\ 4 & i = a + 2 \\ 8 - \epsilon & i = a + 3. \end{cases} \tag{4.3}$$

For example, if k is odd, we have $m'(k) = 8k-5$. Then $a = 2k-2$ and $\epsilon = 3$ in (4.3), and for $i = \langle 2k-1, 2k, 2k+1\rangle$, we have $\nu\binom{32}{i} = \langle 5,4,5\rangle$ and $\nu(|\pi_{4i-1}(P_{8k-5}\wedge bo)|) = \langle 1,4,5\rangle$.

Secondly, we need that $\pi_{4i-1}(P_{m'(k)}) \to \pi_{4i-1}(P_{m'(k)}\wedge bo)$ is injective for $i \le 2k+1$. This is obtained from Tables 8.4, 8.8, 8.14, 8.15, and 8.16 of [7]. These show that for $m'(k) \equiv \langle 3,7,13,14,15\rangle$ mod 16 and $4i-1 \le m'(k) + \langle 8,4,6,5,4\rangle$, the asserted injectivity is true. Now the liftings follow from [3, Theorem 1.1b]. If $k = 16$, the lifting follows for dimensional reasons.

The third thing we need is compatibility. We must show that

$$\begin{array}{ccc} HP^{2k-1} & \longrightarrow & BO_{m'(k-1)} \\ \downarrow & & \downarrow \\ HP^{2k+1} & \longrightarrow & BO_{m'(k)} \end{array}$$

commutes for $k \ge 3$. The two composites agree stably, and so their obstructions to being homotopic lie in $H^*(HP^{2k-1}; \pi_*(V_{m'(k)}))$. If k is even, then $8k-4 < m'(k)$ so the groups are 0. If k is odd, the result follows since $\pi_{8k-4}(V_{8k-5}) = 0$.

We precede the compatible liftings of (4.2) by the canonical maps $RP^{8k+7} \to HP^{2k+1}$, obtaining compatible liftings of $128\xi_{8k+7}$ to $BO_{m(k)}$ for $k \ge 2$. The bundle $128\xi_{15}$ is trivial. To insure compatibility of the liftings on RP^{15} and RP^{23}, we note that the obstructions to compatibility lie in $H^*(RP^{15}; \pi_*(V_{16})) = 0$. This is why we use $m(k) = 16$, rather than 13. ■

Now we prove the induction step. Let

$$\rho(4a+b) = 8a + 2^b \text{ if } 0 \le b \le 3.$$

It satisfies that $2^k\xi_n$ is nontrivial if and only if $n \ge \rho(k)$. Let $\rho = \rho(e-1)$. Assume that we have obtained compatible liftings of $2^{e-1}\xi_{8k+7}$ to $BO_{g(e-1,k)}[\rho]$ for all k. For $0 \le k \le 2^{e-3}$, define

$$g_1(e,k) := \max\{g(e-1,i) + g(e-1,k-i) : \max(0, k-2^{e-4}) \le i \le [k/2]\}.$$

Note that by 2.10.(3),

$$g(e,k) \ge g_1(e,k) - 1. \tag{4.4}$$

Recall $A_k = P^{8k+7}$, and let

$$(A\times A)_k = \bigcup_{i=0}^{k} A_i \times A_{k-i}.$$

Then by 2.8 there are compatible symmetric liftings ℓ_k of $2^{e-1}\xi \times 2^{e-1}\xi$ on $(A \times A)_k$ to $BO_{g_1(e,k)}[\rho]$ for all k. We precede by compatible maps $d_k : A_k \longrightarrow (A \times A)_k$, cellular maps homotopic to the diagonal. The composites $A_k \xrightarrow{\ell_k \circ d_k} BO_{g_1(e,k)}[\rho]$ are compatible liftings of $2^e\xi_{8k+7}$ for all k.

By decreasing induction on k starting with $k = 2^{e-3}$, we will construct compatible factorizations through $BO_{g(e,k)}[\rho]$ of the maps $\ell_k \circ d_k$. Assume inductively that, for all $j > k$, compatible factorizations, up to homotopy rel A_k, of $\ell_j \circ d_j$ through $BO_{g(e,j)}[\rho]$ have been attained. If $g(e,k) \geq g_1(e,k)$, then no factorization of $\ell_k \circ d_k$ is required, and so our induction on k is extended. So we may assume $g(e,k) = g_1(e,k) - 1$.

Let $h = [k/2]$. By 2.10.(4),

$$g_1(e, k-1) \leq g(e,k) - 1. \tag{4.5}$$

By (4.5), 2.10.(4), and the last part of 3.1 (which is required for compatibility of the lifts of $(A \times A)_{k-1}$ and $A_h \times A_h$ to $BO_{g(e,k)-1}$), we have the commutative diagram below, similar to (3.6).

$$\begin{array}{ccccc}
A_{k-1} & \xrightarrow{d'} & (A \times A)_{k-1} \cup A_h \times A_h & \longrightarrow & BO_{g(e,k)-1}[\rho]^{(8k+7)} \\
\downarrow & & \downarrow & & \downarrow \\
A_k & & (A \times A)_k & & BO_{g(e,k)}[\rho]^{(8k+7)} \\
\| & & \| & & \downarrow \\
A_k & \xrightarrow{d_k} & (A \times A)_k & \xrightarrow{\ell_k} & BO_{g(e,k)+1}[\rho]^{(8k+7)} \\
\downarrow & & \downarrow & & {\scriptstyle c}\downarrow \\
A_k/A_{k-1} & \xrightarrow{\bar{d}} & (A \times A)_k/((A \times A)_{k-1} \cup A_h \times A_h) & \xrightarrow{\bar{\ell}} & C,
\end{array}$$

where $C = S^{g(e,k)+1}$ if $g(e,k)$ is odd, and $C = \Sigma P^{g(e,k)}_{g(e,k)-1}$ if $g(e,k)$ is even. The maps labeled d are cellular maps homotopic to the diagonal. The map c is obtained similarly to the first paragraph of the proof of 2.9. Since $\dim(A_k) = 8k+7$, the application of 2.9 requires that

$$8k + 7 \leq \min(\rho + g(e,k) - 1, 2g(e,k) - 1).$$

The second follows from 2.10.(2), while the first follows from $\rho \geq 2e - 2$ and $g(e,k) \geq 8k - e + 2$ since $e \geq 8$.

The quotient $(A \times A)_k/(A_h \times A_h)$ equals $B \vee T(B)$, where T reverses the order of the factors, and B is the union of all cells $e^i \times e^j$ with $i < j$. By the symmetry property of ℓ_k, $\bar\ell|T(B) = (\bar\ell|B)\circ T$. Since $T\circ\bar d \simeq \bar d$, we conclude that $\bar\ell\circ\bar d$ is divisible by 2. Indeed, with r_B denoting the retraction onto B,

$$[\bar\ell\circ\bar d] = [(\bar\ell|B)\circ r_B\circ\bar d] + [(\bar\ell|T(B))\circ r_{T(B)}\circ\bar d]$$

and we have

$$[(\bar\ell|T(B))\circ r_{T(B)}\circ\bar d] = [(\bar\ell|T(B))\circ T\circ r_B\circ\bar d] = [(\bar\ell|B)\circ r_B\circ\bar d].$$

Thus, by 2.9, $\ell_k\circ d_k$ is homotopic rel A_{k-1} to a map which lifts to $BO_{g(e,k)}[\rho]$. Note that the lifting into $BO_{g(e,k)-1}[\rho]$ was not needed if $g(e,k)$ is odd. We have extended our inductive lifting hypothesis, and so have proved that there are compatible liftings of A_k to $BO_{g(e,k)}[\rho]$ for all k. This extends the induction on e and proves 2.10.(6), assuming the first five parts of 2.10.

5 The function $g(e,k)$

In this section, we define the function $g(e,k)$ which has been used in the previous sections, and prove the first five parts of 2.10, its numerical properties which were already used to prove 2.10.(6), its important geometrical property.

We find it convenient to deal with the complementary function G defined by

(5.1) $$G(e,k) = 8k - g(e,k).$$

It has relatively small values, in which patterns are more readily apparent. This function G will be defined using several auxiliary functions.

We define a function S for $k\ge 2$ by

(5.2)

$$S(k) = 8k - 13[\tfrac{k+1}{2}] + 2\alpha(k) + 2\min(3,\nu(k-1)) + \begin{cases} -1 & k\equiv 0\ (2)\\ 2 & k\equiv 1\ (8) \text{ and } \alpha(k)\ne 2\\ 4 & \text{otherwise.}\end{cases}$$

Then $S(k) = 8k - s(k)$, where $s(k)$ is the stable value of $g(e,k)$ when e is sufficiently large. The first values of S are given by

k	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$S(k)$	4	8	7	13	12	16	13	21	18	22	21	27	26	30	25	33	30	34

It will occasionally be useful to set $S(1)=8$, consistent with $g(e,1)=0$.

Values of $k\equiv 1 \bmod 8$ receive special treatment. They are excluded in the domain of some of our functions. For example, for $k\not\equiv 1 \bmod 8$ with $k\geq 2$, we define $V(k)$ by

$$V(k)=\alpha(k)-\begin{cases}2 & k\equiv 3\ (4)\\ 1 & k\not\equiv 3\ (4).\end{cases}$$

The reasons for defining some of these functions will be presented shortly.

We also define functions ν' and R as follows.

$$\nu'(k)=\begin{cases}\nu(k) & k \text{ even}\\ -4 & k \text{ odd},\end{cases}$$

and, for $k\not\equiv 1 \bmod 8$,

(5.3) $$R(k)=S(k)+\nu'(k)-V(k).$$

The first few values of R are given as follows.

k	2	3	4	5	6	7	8	10	11	12	13	14	15	16	18	19
$R(k)$	5	4	9	8	12	11	16	18	17	22	21	25	24	29	30	29

It will also be useful to introduce the notation $\langle n\rangle=\max(0,n)$. We will frequently use the simple fact that for any number X,

(5.4) $$X+\langle -X\rangle\geq 0.$$

Now we can define our function G. An integer k is **decomposable** if it can be written as $k=k_0+\cdots+k_r$ with $r\geq 1$ and $\nu(k_i)>R(k_{i-1})$ for $1\leq i\leq r$. Because each k_i must be preceded in the binary expansion of k by a long string of 0's, it is clear that a decomposable integer has a unique maximal decomposition. The sum in (5.6) is taken over all k_i, $i\geq 1$, in a maximal decomposition of k. The first values of k admitting a decomposition are 35, 66, and 67, with $k_0=3,2,3$, respectively. A simple decomposition is a maximal decomposition with $r=1$. The first value of k admitting a multiple decomposition is $2^{55}+35$ with $k_0=3$ and $k_1=32$.

Definition 5.5 *If $2\leq k\leq 2^{e-3}$ and $(e,k)\neq(7,9)$, we define*

$$G(e,k)=\begin{cases}\min(S(k),G'(e,k)) & k\not\equiv 1\ (8)\\ \min(S(k),6+G'(e,k-1)) & k\equiv 1\ (8),\end{cases}$$

where, for $k \not\equiv 1$ *mod 8,*

$$G'(e,k) = \langle e-6-\nu'(k)\rangle - \sum_{k_i} \langle \min(\nu(k_i), e-6) - R(k_{i-1})\rangle. \tag{5.6}$$

The exceptional value is $G(7,9) = 5$*, not 6 as the formula would give.*

The terms in the sum in (5.6) will sometimes be called *deviations*. We do not define $G(e,0)$, $G(e,1)$, or $G(e,k)$ for $k > 2^{e-3}$; instead we just define the complementary function g by $g(e,0) = g(e,1) = 0$ and $g(e,k) = 2^{e-3}$ for $k > 2^{e-3}$, and observe that the crucial properties (3) and (4) in Theorem 2.10 are easily seen to be satisfied whenever these extreme values are involved.

Next we provide some general discussion of what led to the rather complicated formula for $g(e,k)$. First we describe what led to the basic formula $g(e,k) \approx 8k - \langle e-6-\nu'(k)\rangle$, modified when $k \equiv 1$ mod 8. We began with the initial values $m(k)$ of 4.1 for $g(7,k)$ and used a computer program implementing properties (3) and (4) of 2.10 to obtain bounds for $g(e,k)$ for larger e. Except perhaps for the first few entries in a k-column, the values $8k - (e-6-\nu'(k))$ when $k \not\equiv 1$ mod 8, and $g(e, 8\ell+1) = g(e, 8\ell) + 2$, were apparent until issues of stabilization, which we will discuss shortly, became involved. However, there was no apparent regular pattern for the first few entries in each k-column. The formula $8k - \langle e-6-\nu'(k)\rangle$ was achieved after additional computer experimentation as the simplest general formula satisfying $g(7,k) \geq m(k)$ and consistency with 2.10.

Next we explain where $S(k)$ came from. It is related to the condition $g(e,k) \geq 4k+4$, which says that our lifting methods only work in the stable range. In an earlier version of this paper, we used the triviality of $2^{\phi(n)}\xi_n$ to give 0 as the value of $g(e,k)$ when $e > 4k+3$, but we were unable to prove that this could be done compatibly with our other liftings; i.e. that the liftings which we obtain inductively can be done so that their restrictions to appropriate skeleta are trivial. By forcing $g(7,2) = 16$, we could, as noted in the proof of 4.1, guarantee that our liftings restrict to a trivial bundle on P^{15}, the case $k=1$. For reasons of stability, we forced $g(e,2) \geq 12$ and $g(e,3) \geq 16$. Forcing $g(e,4) \geq 20$ is not strong enough, since, with $g(15,2) = 12$ and $g(15,4) = 25$, we could not obtain $g(16,4) = 24$ consistently with property (4) of 2.10. Thus $g(e,4) = 25$ for $e \geq 15$; i.e. $s(4) = 25$. This translates to our value $S(4) = 8 \cdot 4 - s(4) = 7$.

To be consistent with 2.10, our function S must satisfy the inequalities of the following proposition, the proof of which is straightforward, although somewhat tedious, and is omitted.

Proposition 5.7 *The function S defined in (5.2) satisfies*

$$S(i+j) \le S(i) + S(j) + 1$$

and

$$S(2i) \le 2S(i) - 1,$$

with equality in the first if $i = 2^t$ and $2 \le j \le 2^t - 1$ or $j = 2^t + 1$, and equality holds in the second if $i = 2^t$. Thus S may be defined by $S(2) = 4$, $S(3) = 8$, and

$$S(k) = \min(S(i) + S(k-i) + 1,\ 2 \le i < k/2,\ 2S(k/2) - 1).$$

To be consistent with property (3) of 2.10, our function G must satisfy the property stated in the next theorem, the proof of which will occupy much of this section.

Theorem 5.8 *If $e \ge 8$, and $2 \le i, j \le 2^{e-4}$, then*

$$G(e-1, i) + G(e-1, j) + 1 - G(e, i+j) \ge 0. \tag{5.9}$$

The stabilization given by $S(k)$ and the requirement (5.9) are what lead to the complicated sum in (5.6). The first example of this is for $G(11,3)+G(11,32)+1-G(12,35)$. Since $G'(11,3) = 9 > S(3) = 8$, we have $G(11,3) = 8$. Also $G(11,32) = 11-6-5 = 0$, and $\langle 12 - 6 - \nu'(35)\rangle = 10$. Thus we must subtract 1 from $\langle 12 - 6 - \nu'(35)\rangle$ in $G(12,35)$ in order that (5.9) will hold. This is accounted for by the decomposition of 35 with $k_1 = 3$. The value $R(3) = 4$ is the amount that $\nu(k-3)$ must exceed in order that the decomposition affects the value of $G(e,k)$.

Note that 11 is the smallest value of e for which $G(e,3) \ne \langle e - 6 - \nu'(3)\rangle$. This is obtained by solving

$$e - 6 - \nu'(3) = S(3) + 1,$$

obtaining $e = 11$. We want $R(3)$ to be 1 less than the value of t which satisfies

$$G(11,3) + G(11,2^t) + 1 - \langle 12 - 6 - \nu'(2^t+3)\rangle = -1.$$

Here $G(11,3) - \langle 12 - 6 - \nu'(2^t+3)\rangle$ necessarily equals -2: 1 from $12 - 11$, and 1 from $G(11,3) = G'(11,3) - 1$. Thus we need t to satisfy $0 = G(11,2^t) = 11 - 6 - t$, and so

$$R(3) = t - 1 = (S(3) + \nu'(3) + 6) - 6 = S(3) + \nu'(3),$$

consistent with (5.3), since $V(3) = 0$.

The way V arises can be seen by comparing the requirements, for $t \ge 5$,

$$G(e, 2^t + 5) \le G(e-1, 2) + G(e-1, 2^t+3) + 1$$

and

$$G(e, 2^t+5) \le G(e-1, 5) + G(e-1, 2^t) + 1.$$

The first reduces to, for e moderately large,

$$G(e, 2^t+5) \le S(2) + e - 6 - t + S(3) = e + 6 - t,$$

while the second becomes

$$G(e, 2^t+5) \le S(5) + e - 6 - t = e + 7 - t.$$

We must use the first condition because $S(2) + S(3) < S(5)$. The value $V(5) = 1$ measures this. Our $V(k)$ satisfies that it is the largest r such that $k = i_0 + \cdots + i_r$ with

$$S(i_0 + \cdots + i_t) = S(i_0 + \cdots + i_{t-1}) + S(i_t) + 1$$

for $1 \le t \le r$.

This concludes our discussion of the rationale behind the definition of G except for one more brief comment. It was certainly to be expected that these modifications to the G-formula, given by the summands in (5.6), would be cumulative. It was not *a priori* clear whether $R(k_{i-1})$ or $R(k_0 + \cdots + k_{i-1})$ would be the appropriate part of that formula. The answer will become apparent in Subcase 2d of the proof of 5.11.

The following proposition will be needed shortly. The function S' below will often be encountered in the guise of $S'(k) = R(k) - \nu'(k)$.

Proposition 5.10 *Let* $S'(k) = S(k) - V(k)$. *If* $i, j, i+j \not\equiv 1$ *mod* 8, *then*

$$S'(i) + S'(j) \ge S'(i+j).$$

Moreover, if $i < 2^{\nu(j)}$, *then equality is obtained.*

Proof One easily verifies that

$$S'(k) = 8k - 13[\tfrac{k+1}{2}] + \alpha(k) + \begin{cases} 0 & k \equiv 0\ (2) \\ 8 & k \equiv 3\ (4) \\ 9 & k \equiv 5\ (8). \end{cases}$$

For $1 \le m \le 4$, let ϕ_m denote the mth part of the above formula for $S'(k)$, and let

$\psi_m(i,j)=\phi_m(i)+\phi_m(j)-\phi_m(i+j)$. Then

$$\psi_m(i,j)=\begin{cases}0 & m=1\\ 0 & m=2,\ ij \text{ even}\\ -13 & m=2,\ ij \text{ odd}\\ \nu\binom{i+j}{i} & m=3\\ \geq 16 & m=4,\ ij \text{ odd}\\ -1 & m=4,\ i+j\equiv 5\ (8) \text{ and } i \text{ or } j\equiv 3\ (4)\\ \geq 0 & m=4, \text{otherwise.}\end{cases}$$

Since $\binom{i+j}{i}$ is even if $i+j\equiv 5\ (8)$ and $i\equiv 3\ (4)$, the inequality follows.

For the second part, one easily sees that, if $i<2^{\nu(j)}$, then $\psi_m(i,j)=0$ for $1\leq m\leq 4$. When $m=4$, it is true because $i\equiv i+j$ mod 8 (or $\nu(j)=2$ and $i=2$ or 3). ■

We now begin the lengthy proof of 5.8. In order to keep the number of cases and subcases within reason, we split the theorem into two parts. Most of the work will go into proving the following result.

Theorem 5.11 *If $e\geq 8$, $2\leq i,j\leq 2^{e-4}$, and $i,j,i+j\not\equiv 1$ mod 8, then (5.9) holds.*

Proof We divide into cases depending upon whether $S(i)$ and/or decompositions are involved.

Case 1: Neither i nor j decomposes, $G(e-1,i)\neq S(i)$, and $G(e-1,j)\neq S(j)$. In this case, the LHS of (5.9) becomes

$$\geq \langle e-7-\nu'(i)\rangle+\langle e-7-\nu'(j)\rangle+1-\langle e-6-\nu'(i+j)\rangle. \tag{5.12}$$

By considering separately the four subcases (a) i and j odd, (b) i odd, j even, (c) $\nu(j)>\nu(i)>0$, and (d) $\nu(i)=\nu(j)>0$, one easily shows that (5.12) is ≥ 0 in each subcase. Note that if $i+j$ decomposes, then the LHS of (5.9) is greater than (5.12), and so we need not worry about this possibility here.

Case 2: $G(e-1,i)=S(i)$ and i does not decompose.

Subcase 2a: Also, $G(e-1,j)=S(j)$. Then the LHS of (5.9) is $\geq S(i)+S(j)+1-S(i+j)\geq 0$, by 5.7. The remaining subcases of Case 2 now assume that $G(e-1,j)<S(j)$.

Subcase 2b: j does not decompose, and $\nu(j) \le \nu(i)$. Then $\nu'(i+j) \ge \nu'(j)$, and so the LHS of (5.9) is

$$\ge S(i) + \langle e-7-\nu'(j)\rangle + 1 - \langle e-6-\nu'(i+j)\rangle \ge S(i) > 0.$$

Subcase 2c: j does not decompose, and $\nu(j) > \nu(i)$. We allow for the possibility that i might serve as the bottom part of a decomposition of $i+j$. This will be true if $\nu(j)$ is sufficiently large. Because of our $\langle - \rangle$-notation, our analysis is valid regardless. This time $\nu'(i) = \nu'(i+j)$, and so the LHS of (5.9) is

$$\ge S(i) + \langle e-7-\nu(j)\rangle + 1 - \langle e-6-\nu'(i)\rangle + \langle \min(\nu(j), e-6) - S(i) - \nu'(i) + V(i)\rangle.$$

If $\nu(j) \le e-7$, this is $\ge V(i) \ge 0$. If $\nu(j) \ge e-6$, it simplifies to

$$\ge V(i) + 1 + e - 6 - \nu'(i) - \langle e-6-\nu'(i)\rangle. \tag{5.13}$$

Since $j \le 2^{e-4}$ and $\nu(i) < \nu(j)$, we have $\nu'(i) \le e-5$, and so (5.13) is $\ge V(i) \ge 0$.

Subcase 2d: j admits a decomposition. We consider a 2-stage decomposition $j = j_0 + j_1 + 2^t A$ with A odd and $\nu(j_1) > R(j_0)$. It will be clear that the argument here can be adapted to a longer decomposition. Letting $D \ge 0$ denote any amount added for a decomposition of $i+j$, the LHS of (5.9) becomes, using 5.10,

$$\begin{aligned}
& S(i) + (e-7-\nu'(j)) - (\nu(j_1) - R(j_0)) \\
& \quad -(\min(t, e-7) - R(j_1)) + 1 - \langle e-6-\nu'(i+j)\rangle + D \qquad (5.14)\\
= \ & S'(i) + V(i) + S'(j_0) + S'(j_1) + \nu'(i+j) - \min(t, e-7) + D \\
\ge \ & V(i) + S'(i+j_0+j_1) + \nu'(i+j) - \min(t, e-7) + D \\
= \ & V(i) + R(i+j_0+j_1) - \nu'(i+j_0+j_1) + \nu'(i+j) - \min(t, e-7) + D. \qquad (5.15)
\end{aligned}$$

We will discuss later the removal of the $\langle - \rangle$ at the first step.

We will show below that

$$V(i) - \nu'(i+j_0+j_1) + \nu'(i+j) > 0. \tag{5.16}$$

Assuming this, the only way that (5.15) could be negative is if $\min(t, e-7) > R(i+j_0+j_1)$. But if this is the case, then $(i+j_0+j_1) + 2^t A$ is a decomposition of $i+j$, which makes $D \ge \min(t, e-6) - R(i+j_0+j_1)$. If $i+j_0+j_1$ decomposes further, that only adds more to D. Thus, assuming (5.16), we obtain that (5.15) is ≥ 0.

We now prove (5.16). The only way it could possibly be negative is if $i = 2^t B - j_0 - j_1$ with B even. Then the LHS of (5.16) becomes

$$\begin{aligned}
& \ge \alpha(2^t B - j_0 - j_1) - 2 - (t + \nu(B)) + t \\
& = \alpha(B-1) + t - \alpha(j_0 + j_1 - 1) - 2 - \nu(B) \\
& > 0
\end{aligned}$$

since $\alpha(B-1) \geq \nu(B)$ and $t \geq R(j_0+j_1) >> \alpha(j_0+j_1-1)$.

Regarding the removal of $\langle - \rangle$ above: if $\nu'(i+j) > e-6$, then (5.14) becomes

$$\begin{aligned} &\geq S(i)+e-7-\nu'(j)-\nu(j_1)+R(j_0)-\min(t,e-7)+R(j_1)+1 \\ &= S(i)+(e-7-\min(t,e-7))+(R(j_1)-\nu(j_1))+(R(j_0)-\nu'(j_0))+1 \\ &> 0 \end{aligned}$$

because each of its terms is nonnegative.

Case 3: $G(e-1,i) = S(i)$ and i decomposes. Although the decomposition of i does not affect the value of $S(i)$, it could affect the value of $G(e,i+j)$ by affecting the decomposition of $i+j$. In the analogues of Subcases 2a and 2b, the decomposition of $i+j$ was not needed, and so a decomposition of i cannot affect the validity.

Subcase 3a: j does not decompose and $\nu(j) > \nu(i)$.

Subsubcase 3ai: i admits a simple decomposition. Let $i = i_0 + 2^t\alpha$ with α odd and $t > R(i_0)$. If $\nu(j) \geq t$, then, considering $i_0 + (2^t\alpha) + j$ as a possible decomposition of $i+j$, the LHS of (5.9) becomes

$$\begin{aligned} \geq\ & S(i)+\langle e-7-\nu(j)\rangle+1-\langle e-6-\nu'(i)\rangle \\ & +\langle \min(t,e-6)-R(i_0)\rangle+\langle \min(\nu(j),e-6)-R(2^t\alpha)\rangle. \end{aligned}$$

This exceeds the amount analyzed in Subcase 2c by

$$t-R(i_0)-R(2^t\alpha)+R(i_0+2^t\alpha). \tag{5.17}$$

Since, in the notation of 5.10, $S' = R-\nu'$, and $\nu'(i_0) = \nu'(i_0+2^t\alpha)$, then (5.17) equals $S'(i_0+2^t\alpha)-S'(i_0)-S'(2^t\alpha) = 0$ by 5.10.

If, on the other hand, $\nu(j) < t$, then we don't need $i+j$ to be decomposable, since the LHS of (5.9)

$$\geq S(i)+\langle e-7-\nu(j)\rangle+1-\langle e-6-\nu'(i)\rangle = S(i)+\nu'(i)-\nu(j) > 0,$$

since $S(i) > t+4 > \nu(j)+4$. (The $+4$ is included because of the possibility that $\nu'(i) = -4$.)

Subsubcase 3aii: i admits a multiple decomposition. If $\nu(j) \leq S(i)+\nu'(i)$, then, as in the preceding paragraph, we do not need a decomposition of $i+j$ in order to satisfy (5.9). If, on the other hand, $\nu(j) > S(i)+\nu'(i)$, then the result follows as in the first paragraph of Subcase 3ai, using additivity of S' on disjoint decompositions.

Subcase 3b: i and j both decompose exactly once. Let $i = i_0 + 2^m\beta$ with β odd and $m > R(i_0)$, and $j = j_0 + 2^t\alpha$ with α odd and $t > R(j_0)$.

If $m > t$, then we can consider $i + j$ as $(i_0 + j_0) + 2^t\alpha + 2^m\beta$. It is possible that $\langle m - R(2^t\alpha)\rangle$ might contribute to $G(e, i+j)$, but even if it does, we do not need it. The situation is similar to Subcase 2d. Using the $\langle t - R(j_0)\rangle$ and $\langle t - R(i_0 + j_0)\rangle$ parts of $G(e-1, j)$ and $G(e, i+j)$, respectively, the LHS of (5.9) simplifies to

$$\geq S(i) - \nu'(j_0) + \nu'(i_0 + j_0) + R(j_0) - R(i_0 + j_0),$$

which is very positive. (It would be $\geq V(i_0)$ by 5.10 if $S(i)$ were replaced by the much smaller number $S(i_0)$.) Keeping in mind that $2^{e-3} \geq i + j$, we will usually omit, from now on, explicit consideration of the possibility that $e - 6 < \nu(k_{i-1})$ in (5.6). In Subcase 4d, there is a detailed discussion of a delicate case in which we consider carefully what happens when $e - 6$ is larger than the relevant 2-exponent .

If $m = t$, then a very similar argument works. Because the decomposition of $i + j$ now is $(i_0 + j_0) + 2^p\gamma$ with $p > t$, and this exponent appears with a $+$ sign in $-G(e, i+j)$, the LHS of (5.9) is even larger than it was when $m > t$.

Now suppose $m < t$. We use $(i_0 + j_0 + 2^m\beta) + 2^t\alpha$ as our trial decomposition of $i + j$. If it is not a true decomposition, then the $\langle -\rangle$ will take care of it.

The LHS of (5.9) becomes

$$\begin{aligned}
&\geq && S(i) + (e - 7 - \nu'(j_0)) - (t - R(j_0)) + 1 \\
& && -(e - 6 - \nu'(i_0 + j_0)) + \langle t - R(i_0 + j_0 + 2^m\beta)\rangle \\
&\geq && S(i) - \nu'(j_0) + R(j_0) + \nu'(i_0 + j_0) - R(i + j_0) \\
&= && V(i) + S'(i) + S'(j_0) - S'(i + j_0) \\
&\geq && V(i).
\end{aligned}$$

Subcase 3c: At least one of i and j decomposes more than once. The argument is very similar to that of Subcase 3b. The only reason for separating them is to use 3b as a warmup for 3c. Let $i = i_0 + \cdots + i_r$ and $j = j_0 + \cdots + j_s$ be maximal decompositions.

If $\nu(j_s) \leq \nu(i_r)$, then the LHS of (5.9) is, without using any decomposition of $i + j$,

$$\begin{aligned}
&\geq && S(i) - \nu'(j) - \sum_{k=1}^{s}(\nu(j_k) - R(j_{k-1})) + \nu'(i + j) \\
&\geq && S(i) + \sum_{k=0}^{s-1}(R(j_k) - \nu'(j_k)) - \nu(j_s) \\
&\geq && S(i) - \nu(i_r) \\
&>> && 0.
\end{aligned}$$

If $\nu(i_r) < \nu(j_s)$, first suppose the only decomposition of $i+j$ is the simple decomposition $K + j_s$ with $K = i + j_0 + \cdots + j_{s-1}$. Then the LHS of (5.9) is

$$\begin{aligned} &\geq \quad S(i) - \nu'(j) - \sum_{k=1}^{s}(\nu(j_k) - R(j_{k-1})) + \nu'(i+j) + \nu(j_s) - R(K) \\ &= \quad R(i) + V(i) - \nu'(i) + \sum_{k=0}^{s-1}(R(j_k) - \nu'(j_k)) + \nu'(K) - R(K) \\ &\geq \quad V(i) \end{aligned}$$

by 5.10.

If $i+j$ decomposes more finely, say as $A+B+j_s$, then $-R(K)$ is replaced by $-R(B) + \nu(B) - R(A)$. But these are equal by the second part of 5.10, noting that $\nu'(A+B) = \nu'(A)$.

Case 4: $S(-)$ not involved, i decomposes, j doesn't. Recall $i, j \leq 2^{e-4}$. We assume that i admits a decomposition as $i_0+i_1+i_2$. The nature of our argument will show that the conclusion will also be true for longer decompositions. The LHS of (5.9) becomes

$$e - 6 - \nu'(i_0) - \nu(i_1) + R(i_0) - \nu(i_2) + R(i_1) + \langle e - 7 - \nu'(j)\rangle + Y, \tag{5.18}$$

where $Y = -G(e, i+j)$. We use (5.4) often in what follows.

Subcase 4a: $\nu(j) < \nu(i)$. Then, using a decomposition $i + j = (i_0 + j + i_1) + (i_2)$, we obtain

$$Y \geq -(e - 6 - \nu'(j)) + \langle \nu(i_2) - R(i_0 + j + i_1)\rangle. \tag{5.19}$$

If there is an additional decomposition of $i+j$ as $(i_0+j)+(i_1)+(i_2)$, then by the second part of 5.10, $R(i_0+j+i_1) = R(i_0+j) + R(i_1) - \nu(i_1)$, and so the same expression is obtained. Then (5.18) is

$$\geq (e - 7 - R(i_0 + j + i_1)) + (R(i_0) - \nu'(i_0)) + (R(i_1) - \nu(i_1)) > 0, \tag{5.20}$$

since if the $\langle - \rangle$ in (5.19) is > 0, then

$$e - 7 \geq \nu(i_2) - 2 \geq R(i_0 + j + i_1) - 2,$$

but the $(R - \nu)$-expressions are > 2. If the $\langle - \rangle$ in (5.19) is 0, then the first part of (5.20) is replaced by $(e - 7 - \nu(i_2)) \geq -2$.

Subcase 4b: $\nu(i) \leq \nu(j) < R(i_0)$. In this case, which is very similar to 4a,

$$Y \geq -(e - 6 - \nu'(i)) + \langle \nu(i_2) - R(i_0 + j + i_1)\rangle,$$

because if there is an additional decomposition of $i+j$ as $(i_0+j)+(i_1)+(i_2)$, then $R(i_0+j+i_1)=R(i_0+j)+R(i_1)-\nu(i_1)$, and so the expression for Y is unchanged. Then (5.18) is

$$\geq (S'(i_0)+S'(i_1)+S'(j)-S'(i_0+i_1+j))+(e-7-R(j))>0.$$

In the remaining subcases, we deal with a maximum possible decomposition of $i+j$, realizing, as in 4a and 4b, that if the decomposition must be amalgamated, the expression is not changed.

Subcase 4c: $R(i_0)\leq\nu(j)<\nu(i_1)$. Then

$$Y\geq-(e-6-\nu'(i))+\langle\nu(j)-R(i_0)\rangle+\langle\nu(i_1)-R(j)\rangle+\langle\nu(i_2)-R(i_1)\rangle,$$

and so (5.18) is

$$\geq (e-7-\nu(i_1))+\langle\nu(i_1)-R(j)\rangle>0.$$

Subcase 4d: $\nu(i_1)\leq\nu(j)<\nu(i_2)$. Then

$$Y\geq-(e-6-\nu'(i))+\langle\nu(i_1)-R(i_0)\rangle+\langle\nu(j)-R(i_1)\rangle+\langle\nu(i_2)-R(j)\rangle,$$

and so (5.18) is

$$\geq (e-7-\nu(i_2))+\langle\nu(i_2)-R(j)\rangle>0. \tag{5.21}$$

As noted in Subcase 3b, we are usually not paying explicit attention to the possibility that $e-6\leq\nu(i_2)$ (in the situation in this subcase, 4d). Here it does warrant our attention. We might have $i_2=2^{e-5}$, 2^{e-6}, or $3\cdot2^{e-6}$, and then it would seem that (5.21) might not be valid.

If $i_2=2^{e-5}$, then $\langle\nu(i_2)-R(i_1)\rangle$ in the above analysis is replaced by $\langle e-7-R(i_1)\rangle$. This decrease of 2 compensates for the fact that $e-7-\nu(i_2)=-2$ in (5.21). Similarly, if $\nu(i_2)=e-6$, then $\langle\nu(i_2)-R(i_1)\rangle$ is replaced by $\langle e-7-R(i_1)\rangle$, compensating for $e-7-\nu(i_2)=-1$.

Subcase 4e: $\nu(i_2)<\nu(j)$. Then

$$Y\geq-(e-6-\nu'(i))+\langle\nu(i_1)-R(i_0)\rangle+\langle\nu(i_2)-R(i_1)\rangle+\langle\nu(j)-R(i_2)\rangle,$$

and so (5.18) is

$$\geq\langle e-7-\nu(j)\rangle+\langle\nu(j)-R(i_2)\rangle>0.$$

Case 5: $S(-)$ not involved, both i and j decompose. We consider here a typical example in which both i and j decompose twice. It should be clear that the general case will work out in the same way. We assume that $i = i_0 + i_1 + i_2$ and $j = j_0 + j_1 + j_2$ are decompositions. Then

$$\begin{aligned} G(e-1,i)+G(e-1,j)+1 &= e-6-\nu'(i_0)-\nu(i_1)+R(i_0)-\nu(i_2)+R(i_1) \\ &\quad +e-7-\nu'(j_0)-\nu(j_1)+R(j_0)-\nu(j_2)+R(j_1) \end{aligned}$$

We assume without much loss of generality that $\nu(j_2) > \nu(i_2)$ and $\nu(i_0) < \nu(j_0)$.

Subcase 5a: $\nu(j_2) < R(i_0+i_1+i_2+j_0+j_1)$. We use no decomposition of $i+j$. We obtain that

$$\begin{aligned} & G(e-1,i)+G(e-1,j)+1-G(e,i+j) \\ \geq\ & R(i_0)+S'(i_1)-\nu(i_2)+e-7+S'(j_0)+S'(j_1)-\nu(j_2) \\ =\ & S'(i_0)+S'(i_1)+S'(i_2)+S'(j_0)+S'(j_1)+\nu'(i_0)-R(i_2)+e-7-\nu(j_2) \\ \geq\ & R(i_0+i_1+i_2+j_0+j_1)-R(i_2)+e-7-\nu(j_2) \\ >>\ & 0, \end{aligned}$$

since $e-7-\nu(j_2) \geq -2$ while $R(i_0+i_1+i_2+j_0+j_1)-R(i_2) >> 0$.

Subcase 5b: $\nu(j_2) > R(i_0+i_1+i_2+j_0+j_1)$. We use a decomposition of $i+j$ as $(i_0+i_1+i_2+j_0+j_1)+(j_2)$. We discuss afterward the usual argument regarding what happens if it decomposes more finely. Similarly to Subcase 5a, we obtain

$$\begin{aligned} & G(e-1,i)+G(e-1,j)+1-G(e,i+j) \\ \geq\ & S'(i_0)+S'(i_1)+S'(i_2)+S'(j_0)+S'(j_1)-S'(i_0+i_1+i_2+j_0+j_1) \\ & +e-7-R(i_2) \\ \geq\ & 0 \end{aligned}$$

using 5.10 and

$$e-7 \geq \nu(j_2)-2 \geq R(i_0+i_1+i_2+j_0+j_1)-2 >> R(i_2).$$

Further decomposition of $i_0+i_1+i_2+j_0+j_1$ into 2-adically disjoint parts does not change the expression, using the second part of 5.10, similarly to the argument in Subcases 4a and 4b. ■

The following result will be useful in some subsequent proofs. In particular, 2.10.(5) is an immediate consequence.

Proposition 5.22 *For* $e \geq 7$ *and* $2 \leq k < 2^{e-3}$*,*

$$G(e,k+1)-G(e,k)\begin{cases} =8 & k\equiv 0\ (8),\ \alpha(k)=1,\ e\geq S(k)+\nu(k)+8\\ =7 & k\equiv 0\ (8),\ \alpha(k)=1,\ e=S(k)+\nu(k)+7\\ =6 & k\equiv 0\ (8),\ \textit{otherwise}\\ \leq -1 & k\equiv 1\ (8)\\ \leq 6 & \textit{otherwise.}\end{cases}$$

Proof We begin by noting that the result is true for the limiting values, $S(k)$, since they are easily shown to satisfy

$$S(k+1)-S(k)\begin{cases} =8 & k=2^e,\ e\geq 3\\ =6 & k\equiv 0\ (8),\ \alpha(k)>1\\ =6 & k\equiv 4\ (8)\\ =4 & k\equiv 2\ (4)\\ \leq -1 & k\equiv 1\ (8)\\ =-1 & k\equiv 3,5\ (8)\\ \leq -3 & k\equiv 7\ (8).\end{cases}\tag{5.23}$$

The case $k \equiv 0 \bmod 8$ of the proposition follows easily from (5.23) and the definitions.

We next handle the case $k = 8\ell+1$. If $\nu(\ell) \geq 3$, then $8\ell+2$ admits a decomposition with $k_0 = 2$. Any additional portions of a decomposition of $8\ell+2$ will occur identically in 8ℓ. Thus, in this case, with $\nu = \nu(8\ell) \geq 6$,

$$G(e,8\ell+2)-G(e,8\ell+1) = e-7-\langle \min(\nu,e-6)-5\rangle-(6+\langle e-6-\nu\rangle).$$

This is ≤ -2, regardless of the sign of $e-6-\nu$.

Now assume $\nu(\ell) < 3$. If 8ℓ admits a decomposition as $k_0 + 2^t\alpha$ with α odd, then we consider $(k_0+2)+2^t\alpha$ as a possible decomposition of $8\ell+2$. Any additional portions of a decomposition of $8\ell+2$ occur identically in 8ℓ. For $v = \nu(\ell) = 0$, 1, or 2, we obtain

$$G(e,8\ell+2)-G(e,8\ell+1) = e-13-\langle e-9-v\rangle-\langle D-2+v\rangle+\langle D\rangle,\tag{5.24}$$

where $D = \min(t,e-6)-R(k_0)$. Here we have used the easily-verified fact that if $k_0 \equiv 0 \bmod 8$, then $R(k_0+2)-R(k_0) = 5-\nu(k_0)$. One easily checks that (5.24) is ≤ -2 for any e and D, since $0 \leq v \leq 2$.

For $\tau = [2,3,4,5,6,7]$ and $k = 8\ell+\tau$, we have, for $e > 7$,

$$\langle e-6-\nu'(k+1)\rangle - \langle e-6-\nu'(k)\rangle = [5,-6,6,-5,5,\leq -5],$$

and, if k admits a simple decomposition $k_0 + 2^t\alpha$ with α odd,

$$\langle m - R(k_0)\rangle - \langle m - R(k_0+1)\rangle \le [0,5,0,4,0,5].$$

Here $m = \min(e-6, t)$. As before, higher deviations will cancel in the difference. Thus $G(e,k+1) - G(e,k)$, which is the sum of the two displays of this paragraph, is ≤ 6, as claimed. ∎

Now we can complete the proof of 5.8 by proving.

Theorem 5.25 *Theorem 5.8 is true when i or j or $i+j$ is $\equiv 1 \bmod 8$.*

Proof Again we divide into cases.

Case 1: Only $i \equiv 1$ mod 8. We have

$$\begin{aligned} & G(e-1,i) + G(e-1,j) + 1 - G(e,i+j) \\ = \; & \big(G(e-1,i) - G(e-1,i-1)\big) - \big(G(e,i+j) - G(e,i-1+j)\big) \\ & + \big(G(e-1,i-1) + G(e-1,j) + 1 - G(e,i-1+j)\big) \\ \ge \; & 0, \end{aligned} \tag{5.26}$$

since the first $(-)$ in (5.26) is ≥ 6 by 5.22, the second is ≤ 6 by 5.22, and the third is ≥ 0 by 5.11.

Case 2: both i and $j \equiv 1$ mod 8. This follows by an argument similar to that of Case 1.

Case 3: i and $i+j \equiv 1$ mod 8. This follows from the validity for $(i-1,j)$ similarly to Case 1. Usually $G(e-1,i) - G(e-1,i-1) = 6$ and $G(e,i+j) - G(e,i-1+j) = 6$, and so the inequality follows as in (5.26). If $G(e,i+j) - G(e,i-1+j) > 6$, then $G(e-1,i) = S(i)$ and $G(e-1,j) = S(j)$, and so

$$G(e-1,i) + G(e-1,j) + 1 - G(e,i+j) \ge S(i) + S(j) + 1 - S(i+j) \ge 0$$

by 5.7.

Case 4: $i+j \equiv 1$ mod 8, while $i,j \not\equiv 1$ mod 8. If $G(e,i+j) - G(e,i+j-1) > 6$, then $G(e-1,i) = S(i)$, $G(e-1,j) = S(j)$, and $G(e,i+j) \le S(i+j)$, and so the result follows from 5.7. So we may now assume $G(e,i+j) - G(e,i+j-1) = 6$. Without loss of generality, assume i is odd and j is even.

First, we assume $i \equiv 3 \bmod 4$. By the proof of 5.22, $G(e,i)-G(e,i-1)=4$ or 5, and if i is indecomposable, then $G(e,i)-G(e,i-1)=4$ if and only if $G(e,i-1)=S(i-1)$. Thus the result will follow as in (5.26) once we show that if $i,j \equiv 2 \bmod 4$ and $i+j \equiv 0 \bmod 8$, then (5.9) is satisfied with 1 to spare, and with 2 to spare if $G(e,i)-G(e,i-1)=4$.

The basic value of the LHS of (5.9) in this case is

$$\langle e-8\rangle + \langle e-8\rangle + 1 - \langle e-v\rangle \tag{5.27}$$

with $v \geq 9$. This equals 1 if $e=7$ or 8, while for $e \geq 9$, it is $\geq e-6$. The smallest e for which the LHS of (5.9) does not equal (5.27) is $e=12$, when $i=2$.

Neglecting temporarily the effect of deviations, the desired conclusion is obtained since it is true at the onset of $S(i)$ and will continue to be true as e increases, since now $G(e-1,j)$ and $G(e,i+j)$ will both increase by 1 each time. When $G(e-1,j)$ achieves a value of $S(j)$, then the LHS of (5.9) is

$$\geq S(i)+S(j)+1-S(i+j) > 2$$

for the congruences being considered here.

When deviations are taken into account, the fact that makes it work is the easily-verified fact that

$$R(8\ell+2)+R(8\ell'+6)-R(8\ell+8\ell'+8) = 1+\nu\big(\tbinom{\ell+\ell'}{\ell}\big). \tag{5.28}$$

Suppose, for example, that $i=i_0+2^t\alpha$ and $j=j_0+2^u\beta$ are decompositions with α and β odd, and $t<u\leq e-7$. The LHS of (5.9) becomes

$$\geq e-8+e-7-(t-R(i_0))-(u-R(j_0))-(e-v)+\langle t-R(i_0+j_0)\rangle+\langle u-R(2^t\alpha)\rangle$$

with $v \geq 9$. Using (5.28), this is

$$\begin{aligned} &\geq\ e+v-14+R(i_0+j_0)-t+\nu\big(\tbinom{i_0+j_0}{i_0}\big)+\langle t-R(i_0+j_0)\rangle-u\\ &\geq\ v-7\\ &\geq\ 2, \end{aligned}$$

since $e-7\geq u$ and using (5.4). Other situations involving decompositions work out similarly.

The case $i \equiv 5$ is handled similarly. ■

Next we verify the first part of 2.10.(4). In fact the conclusion of that theorem is true without regard for the hypothesis.

Theorem 5.29 *If $i, j \le 2^{e-3}$ and $i+j+1 \le 2^{e-2}$, then*

$$g(e,i) + g(e,j) < g(e+1, i+j+1).$$

Proof We prove the equivalent statement, with i, j, and e as in the hypothesis,

$$G(e,i) + G(e,j) + 8 > G(e+1, i+j+1). \tag{5.30}$$

By 5.8 and 5.22, we have

$$G(e,i) + G(e,j) + 8 \ge G(e+1, i+j) + 7 > G(e+1, i+j+1)$$

unless $i+j+1 = 2^t + 1$ with $t \ge 3$ and $G(e+2, i+j+1) = S(i+j+1)$. In this case, it will also be true that $G(e,i) = S(i)$ and $G(e,j) = S(j)$. Thus it suffices to show

$$S(i) + S(2^t - i) + 8 > S(2^t + 1).$$

This follows readily from the definition of S. The smallest value of $S(i) + S(2^t - i)$ occurs when $i = 2^{t-1}$ and is $3 \cdot 2^{t-1} + 2$, while $S(2^t + 1) = 3 \cdot 2^{t-1} + 9$. ■

The second part of 2.10.(4) follows from the following result.

Theorem 5.31 *For $k \le 2^{e-3}$, $G(e+1, 2k) \le 2G(e,k)$ with equality if and only if $G(e+1, 2k) = G(e,k) = 0$, which occurs if and only if*

$$k \in \{2^{e-3},\ 2^{e-4},\ 2^{e-5},\ 3 \cdot 2^{e-5},\ 2^{e-6}\alpha\} \text{ with } \alpha \in \{1,3,5,7\}.$$

If equality occurs, then

$$G(e+1, 2k) < G(e, \ell) + G(e, 2k - \ell) + 1$$

for all ℓ.

Proof The second sentence follows immediately from the first, since

$$0 < G(e, \ell) + G(e, 2k - \ell) + 1.$$

For basic values, we have

$$2G(e,k) - G(e+1, 2k) = \begin{cases} 2(e-2) - (e-6) & k \text{ odd} \\ \langle e - 6 - \nu(k) \rangle & k \text{ even.} \end{cases}$$

This is clearly ≥ 0, and $= 0$ in exactly the cases claimed.

If $G(e,k)=S(k)$, then

$$2G(e,k)-G(e+1,2k)\geq 2S(k)-S(2k)=\begin{cases}2\alpha(k)-1 & k \text{ even}\\ 12 & k=2^t+1,\ t\geq 3\\ 2\alpha(k)+4 & k\equiv 1\ (8),\ \alpha(k)\neq 2\\ 4\nu(k-1)+2\alpha(k)-4 & k\equiv 3,5,7\ (8).\end{cases}$$

This is >0.

Suppose $k=k_0+2^t\alpha$ is a simple decomposition, with α odd and $e-6\geq t$. If k is even, then $2R(k)=R(2k)+\alpha(k-1)$, and so

$$\begin{aligned}&2G(e,k)-G(e+1,2k)\\ =\ & e-6-\nu(k)-2(t-R(k_0))+\langle t+1-R(2k_0)\rangle\\ =\ & e-5-t+\alpha(k_0)-1+R(2k_0)-t-1+\langle t+1-R(2k_0)\rangle\\ \geq\ & 1,\end{aligned}$$

using (5.4). If $k\equiv 3,5,7$ mod 8, then

$$2R(k)=R(2k)+4\nu(k-1)+\alpha(k)-\begin{cases}10 & k\equiv 3\ (4)\\ 12 & k\equiv 1\ (4).\end{cases}$$

Then

$$\begin{aligned}&2G(e,k)-G(e+1,2k)\\ =\ & e+2-2(t-R(k_0))+\langle t+1-R(2k_0)\rangle\\ \geq\ & e+3-t+R(2k_0)-t-1+\langle t+1-R(2k_0)\rangle+4\nu(k_0-1)+\alpha(k_0)-12\\ >\ & 0.\end{aligned}$$

The situation when $t>e-6$ and the case of higher deviations are handled similarly.

Finally, we have

$$\begin{aligned}&2G(e,8\ell+1)-G(e+1,16\ell+2)\\ \geq\ & 2(G(e,8\ell)+6)-G(e+1,16\ell)-(G(e+1,16\ell+1)-G(e+1,16\ell))\\ & -(G(e+1,16\ell+2)-G(e+1,16\ell+1))\\ \geq\ & 12+0-8-(-1)\\ >\ & 0.\end{aligned}$$

■

Finally, we verify part (2) of 2.10. We have

$$g(e,k) = 8k - G(e,k) \geq 8k - S(k) \geq 13[\tfrac{k+1}{2}] - 2\alpha(k) - 10.$$

This is $\geq 4k+4$ for $k \geq 7$, while for $k < 7$ we verify directly that $8k - S(k) \geq 4k+4$.

6 A bound for geometric dimension of normal bundle

In this section, we prove the following key result, a main ingredient in the proof of our geometric dimension result, 2.11, which has already been seen to imply our immersion theorem.

Theorem 6.1 *If $e \geq 7$ and $t \geq 1$ and $k_0 + \cdots + k_{t-1} = 2^{e-3} - 1$, then*

$$\sum_{i=0}^{t-1} G(e+i, k_i) \geq e - 2. \tag{6.2}$$

Remark 6.3 The integers k_i in this theorem are nonnegative, but possibly zero. Some examples in which equality is obtained are

- $G(e, 2^{e-3} - 1)$;
- $G(e, 2^{e-4} - 1) + G(e+1, 2^{e-4})$;
- $G(e, 2^{e-5} - 1) + G(e+1, 3 \cdot 2^{e-5})$;
- $G(e, 3 \cdot 2^{e-5} - 1) + G(e+1, 2^{e-5})$;
- $G(e, 2^{e-5} - 1) + G(e+1, 2^{e-5}) + G(e+2, 2^{e-4})$;
- $G(e, 2^{e-4} - 1) + G(e+1, 0) + G(e+2, 2^{e-4})$.

Before proving the theorem, we provide the easy deduction of 2.11.

Proof of 2.11 From 6.1 and (5.1), we obtain

$$\sum_{i=0}^{t-1} g(e+i, k_i) \leq (2^e - 8) - (e-2) = 2^e - e - 6. \tag{6.4}$$

Let e be fixed, and for $t \geq 1$ and $0 \leq \ell \leq 2^{e-3} - 1$, let

$$M(t,\ell) = \max\left(\sum_{i=0}^{t-1} g(e+i, k_i) : \ k_0 + \cdots + k_{t-1} = \ell \right).$$

Then $M(t,\ell)=\max(M(t-1,i)+g(e+t-1,\ell-i):\ 0\le i\le\ell)$. Using 2.8, induction on t, and 2.10.(6), we obtain that for all t and $\ell\le 2^{e-3}-1$

$$\mathrm{gd}((2^e+\cdots+2^{e+t-1}),\ell)\le M(t,\ell)$$

compatibly for all ℓ. By (6.4), $M(t,2^{e-3}-1)\le 2^e-e-6$. Since $\mathrm{gd}(n,k)=\mathrm{gd}(n\xi_{8k+7})$, we obtain the conclusion of 2.11. ■

The proof of 6.1 is expedited by the following lemma.

Lemma 6.5 *Let $d>0$. If $G(e,i)<S(i)$ and $G(e+d,j)<S(j)$, then*

$$G(e,i)+G(e+d,j)\ge G(e,i+j).$$

Proof This follows exactly as in the proofs of Cases 1, 4, and 5 of 5.11 and the proof of 5.25. In those results, there was an extra 1 on the LHS, but the larger e-components here more than compensate for that. ■

Remark 6.6 Lemma 6.5 is not always true when $S(-)$ is involved. For example, if $e\ge 15$, then $G(e,2)+G(e+1,3)=12<13=G(e,5)$.

Proof of 6.1 Let $\mathcal{S}$ denote the set of those k_i for which $G(e+i,k_i)=S(k_i)$. This includes cases in which $k_i=0$ or $k_i=1$. If $\mathcal{S}$ is empty, then the result follows by induction from 6.5, since $G(e,2^{e-3}-1)=e-2$. Let $K=\sum_{k_i\in\mathcal{S}}k_i$. We split the LHS of (6.2) as

$$\sum_{k_i\in\mathcal{S}}G(e+i,k_i)+\sum_{k_i\notin\mathcal{S}}G(e+i,k_i). \tag{6.7}$$

Since, as is easily proved, $S(k)\ge\frac{3}{2}k$, the first half of (6.7) is $\ge\frac{3}{2}K$, while 6.5 implies that the second half of (6.7) is

$$\ge G(e,2^{e-3}-1-K)\ge e-6-\nu'(K+1)-D(e,2^{e-3}-1-K),$$

where $D(-,-)$ denotes the deviation, i.e., the sum in (5.6). Now the desired inequality reduces to

$$\tfrac{3}{2}K\ge\nu'(K+1)+4+D(e,2^{e-3}-1-K). \tag{6.8}$$

If $K \neq 1,3$, this inequality is true, usually with much to spare. Indeed, $K \geq \nu'(K+1)+4$ if $K \neq 1,3$, and

$$\tfrac{1}{2}K \geq D(e, 2^{e-3}-1-K). \tag{6.9}$$

To see (6.9), note that for $D(e,k)$ to be positive due to a single deviation, then $k = 2^t\alpha + k_0$ with $t > R(k_0) > k_0$, α odd, and $D(e,k) = t - R(k_0)$. For such k, if $k = 2^{e-3}-1-K$, then $K \geq 2^t - 1 - k_0$, and so the difference in (6.9) is

$$\geq \tfrac{1}{2}(2^t-1-k_0) - (t - R(k_0)) = (\tfrac{1}{2}(2^t-1)-t) + (R(k_0) - \tfrac{1}{2}k_0) > 0,$$

and a similar analysis applies when multiple deviations are involved. When $K = 1,3$, (6.8) is true if the LHS is replaced by $S(K) = 8$. ■

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