2024 LEHIGH UNIVERSITY HIGH SCHOOL MATH CONTEST

1. (2 pts) How many positive integers are divisors of 2024?
2. (2 pts) What is the smallest positive integer x such that $2024-x$ is the cube of an integer?
3. (2 pts) A square with sides of length 4 is partitioned into four congruent subsquares. A circle passes through the centers of the four subsquares. What is the area of the circle?
4. (2 pts) In a certain football league the only possible scores are 7 for a touchdown and 3 for a field goal. Nothing else is possible, so it is not like ordinary football. How many positive scores are not possible?
5. (2 pts) You flip a coin five times and heads occurs three times. Given this information, what is the probability that the first two flips were heads?
6. (2 pts) Find all ordered triples (a, b, c) which satisfy
$$a^2 + 2b^2 - 2bc = 100 \text{ and } 2ab - c^2 = 100.$$
7. (2 pts) Suppose $a > b > 0$ and $\frac{a}{b} + \frac{b}{a} = 6$. Evaluate $\left(\frac{a+b}{a-b}\right)^3$.
8. (2 pts) Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of the real number x. List all real numbers x such that $\lfloor x \rfloor^2 = x\{x\} + \{x\}^2$.
9. (3 pts) Let $f(x) = x^2 - 2x + t$. Find all real numbers t such that for all $a, b, c \in [0, 3]$ (not necessarily distinct) there exists a triangle with positive area with side lengths $f(a)$, $f(b)$, and $f(c)$.
10. (3 pts) Let $f(x) = \frac{1}{2} + \log_{10}\left(\frac{x}{1-x}\right)$. What is the value of $\sum_{i=1}^{99} f(\frac{i}{100})$?

11. (3 pts) A circle inscribed in triangle ABC touches AB at D with $AD = 5$ and $DB = 3$. Find the length of BC if angle A is 60 degrees.

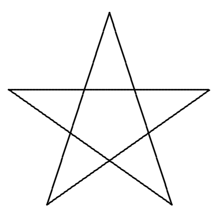
12. (3 pts) Find all real numbers t such that there are exactly five integers n satisfying the inequalities below. (You may express your answer using interval notation or inequalities.)
$$\frac{2n+5}{3} - n > -5, \qquad \frac{n+3}{2} - t < n.$$

13. (3 pts) How many ordered pairs of integers (m, n) satisfy the equation $\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$? Note that negative integers are allowed, as well as positive.

14. (3 pts) An ant is moving around the vertices of a cube. Each move takes it from from its current vertex to one of the three adjacent vertices. What is the probability that after four moves it is back where it started?

15. (3 pts) Suppose functions f and g satisfy that for all real numbers x and y, $f(x + g(y)) = 2x + y + 5$. Find an explicit expression for $g(x + f(y))$.

16. (3 pts) A regular 5-pointed star (equal angles at each tip and equal lines connecting tips) has total length of the 5 lines connecting the tips equal to 1. What is the perimeter of the regular pentagon in its interior? Your answer should involve a specific trig function and specific angle, expressed in degrees.



17. (3 pts) There is a pot of candies and a group of people who want them. Each person gets only one chance and must take exactly 1/3 or 1/2 of the remaining candy. What is the largest number of people who could take candies from some initial amount less than 1000? Fractional candies are not allowed. The initial amount is chosen to accommodate the maximum number of people.

18. (3 pts) Find all values of k such that the equation

$$4x^2 + 4(2-k)x - k^2 = 0$$

has two real solutions x_1 and x_2 satisfying $|x_1| = 2 + |x_2|$.

19 (4 pts) If n is a positive integer, let $p(n)$ be the product of the digits of n, and let $r(n)$ be the integer obtained by reversing the order of the digits of n. For example, $r(125) = 521$. Find all positive integers n such that $p(n) > 0$ and $n \cdot r(n) = 1000 + p(n)$.

20. (4 pts) Let x_{400} be 400th smallest positive solution of the equation $x - \frac{\pi}{2} = \tan x$. Find the greatest integer that does not exceed $\frac{1}{2}x_{400}$.

21. (4 pts) Find the minimum value of $(3\cos\alpha + 5\sin\beta - 8)^2 + (3\sin\alpha + 5\cos\beta - 15)^2$ for all real numbers α and β.

22. (4 pts) Let ABC be a triangle with $AB = 9$, $BC = 10$, and $CA = 17$. Let B' be the reflection of the point B across the line AC. Let G (resp. G') be the centroid of triangle ABC (resp. $AB'C$). What is the length of GG'?

23. (4 pts) Suppose $a > 1$ and $x, y > 0$ satisfy

$$(\log_a x)^2 + (\log_a y)^2 - \log_a(x^2y^2) \leq 2$$

and $\log_a y \geq 1$. Find the interval of values of $\log_a(x^2y)$.

24. (4 pts) Find the number of integers n such that n is a multiple of 5, and the base-11 expansion of n, $\overline{def}_{11}$, satisfies $0 < d < f$.

25. (5 pts) Triangle ABC has $AB = 4$, $BC = 5$, and $CA = 6$. The circumcircle of ABC is also the incircle of triangle $A'B'C'$, with A on $B'C'$, B on $A'C'$, and C on $A'B'$. What is the length of $B'C'$?

26. (5 pts) Suppose f and g are inverse functions. Let $F(x) = f(x+1) - 2$ and $G(x) = g(2x+1)$, and suppose that F and G are also inverse functions. Finally, suppose $f(1) = 4$. Find $f(100)$.

27. (5 pts) 27 balls, numbered 1 through 27, are placed in three bowls labeled A, B, and C with at least four balls in each bowl. If the averages of the numbers on the balls in bowls A, B, and C are 15, 3, and 18, respectively, what is the largest possible number on the ball with the smallest number in bowl A?

28. (5 pts) Let $r_1, \ldots, r_7$ denote the distinct roots of the polynomial $x^7 - 5$. What is the value of $\prod_{1 \le i < j \le 7} (r_i + r_j)^2$?

29. (5 pts) Find all 5-digit numbers $\overline{abcde}$, not containing the digit 0, such that $e \mid \overline{de} \mid \overline{cde} \mid \overline{bcde} \mid \overline{abcde}$. That is, repeatedly removing the left digit leaves a number which is a divisor of the previous number. The digits need not be distinct.

30. (5 pts) Cards numbered 1 to 20 are placed randomly in slots numbered 1 to 20, with one card in each slot. A *transposition* consists of interchanging the slots of two cards whose numbers have the opposite parity, leaving other cards fixed. How many transpositions are required to get every card into the slot with the same number from any initial arrangement?

SOLUTIONS. (annotated with number of correct responses out of 75)

1. 16. [60] $2024 = 2^3 \cdot 11 \cdot 23$, so it has $(3+1)(1+1)(1+1)$ positive divisors.
2. 296. [71] Since 2024 lies between $12^3 = 1728$ and $13^3 = 2197$, the answer is $x = 2024 - 1728$.
3. 2π. [69] Put the vertices of the square at $(\pm 2, \pm 2)$. The circle is centered at $(0,0)$ and passes through $(\pm 1, \pm 1)$, so its radius is $\sqrt{2}$.
4. 6. [61] They are 1, 2, 4, 5, 8, and 11. It is easy to see that other scores can be obtained. For example, $13 = 7 + 3 + 3$.
5. 3/10. [67] There are $\binom{5}{3} = 10$ possibilities for the flips on which the H's occurred. Three of them started HH, with the final H occurring on any one of the final three flips.
6. $(10, 10, 10)$ and $(-10, -10, -10)$. [40] Subtracting the equations yields $a^2 + 2b^2 + c^2 - 2ab - 2bc = 0$, which says $(a-b)^2 + (c-b)^2 = 0$, so $a = b = c$ and $a^2 = 100$.
7. $2\sqrt{2}$ or $2^{3/2}$. [49.5] We obtain $a^2 + b^2 = 6ab$, so $(a+b)^2 = 8ab$ and $(a-b)^2 = 4ab$. Thus $\left(\frac{a+b}{a-b}\right)^2 = 2$. So $\left(\frac{a+b}{a-b}\right)^3$ is as claimed.
8. 0, 1.5. [41] Let $\lfloor x \rfloor = n$ and $\{x\} = t$, so $x = n + t$. Then $n^2 = (n+t)t + t^2$, so $(n - 2t)(n + t) = 0$. The second factor is 0 when $x = 0$. The first factor is 0 if $t = .5$ and $n = 1$.
9. $t > 5$. [31] The minimum value for f is $f(1) = t - 1$. The maximum value of f on $[0, 3]$ is $f(3) = 3 + t$. The longest side must be less than the sum of the two shorter sides, so we must have $3 + t < 2(t - 1)$. (The borderline case, $t = 5$, can yield a "triangle" with sides 4, 4, and 8 if $a = b = 1$, but this does not have positive area.)
10. 99/2. [49] Note that $f(x) + f(1-x) = 1$ for all $x \in (0, 1)$. Thus
$$2\sum_{i=1}^{99} f\left(\tfrac{i}{100}\right) = \sum_{i=1}^{99}\left(f\left(\tfrac{i}{100}\right) + f\left(1 - \tfrac{i}{100}\right)\right) = 99.$$

11. 13. [37] Let E and F be the points of tangency on BC and AC, and $x = CE = CF$. The Law of Cosines on triangle ABC yields $(x+3)^2 = 8^2 + (x+5)^2 - 8(x+5)$. This equation has solution $x = 10$, so $BC = 13$.

12. $-6 < t \leq -\frac{11}{2}$ or $(-6, -\frac{11}{2}]$. [36] The first inequality says $n < 20$, while the second says $n > 3 - 2t$. To have exactly five integer solutions for n, we must have $14 \leq 3 - 2t < 15$.

13. 5. [41] It reduces to $2n + 2m = mn$ or $(m-2)(n-2) = 4$. Thus $(m-2, n-2)$ can be $(2,2)$, $\pm(4,1)$, or $\pm(1,4)$. Note that $(-2,-2)$ does not work since it would have $m = n = 0$.

14. 7/27. [40.5] Let the cube have vertices at 0 or 1 in each coordinate, and the ant start at (0,0,0). After 3 moves, it is at (1,1,1) with probability $3!/27 = 2/9$, and otherwise is at one of (1,0,0), (0,1,0), or (0,0,1). In the first case, it cannot get back to (0,0,0), while in the latter cases, it gets back with probability 1/3. Thus the answer is $\frac{7}{9} \cdot \frac{1}{3}$.

15. $\frac{1}{2}x + y + \frac{5}{2}$. [43] You can probably guess that f and g will be linear. (The problem can be solved without that assumption.) Let $f(x) = ax + b$ and $g(x) = cx + d$. Then $a(x + cy + d) + b$ is equal to $2x + y + 5$, which implies that $a = 2$, $c = \frac{1}{2}$, and $ad + b = 5$. The desired function is $c(x + ay + b) + d$, which is $\frac{1}{2}x + y + \frac{5}{2}$.

16. $\sin(18)/(1 + \sin(18))$ or $1 - \frac{1}{1+\sin(18)}$. [30] Let x denote the exterior segments and y the sides of the pentagon. Since the angles in the pentagon are 108, the angles at the tip are $180 - 2 \cdot 72 = 36$. Thus $\frac{y/2}{x} = \sin(18)$ and $2x + y = 1/5$. These simplify to $y = (\frac{1}{5} - y)\sin(18)$, leading to the answer for the perimeter $5y$. (Many other forms of the answer were received and credited. We always checked the decimal value.)

17. 12. [45] Suppose p people take 1/3 and q people take 1/2. We want the largest value of $p+q$ such that $(\frac{3}{2})^p 2^q < 1000$ and $p \le q$, so that the number of candies is always an integer. With $p = q = 6$, we could start with 729. Increasing either p or q by 1 multiplies 729 by 3/2 or 2, putting it above 1000. Note that 729 works by alternately taking 1/3, then 1/2, then 1/3, etc.

18. 0 and 4. [16] Note that $x_1x_2 = -k^2/4$ and $x_1 + x_2 = k - 2$. We have

$$\begin{aligned} 4 &= (|x_1| - |x_2|)^2 = x_1^2 + x_2^2 - 2|x_1x_2| \\ &= (x_1 + x_2)^2 - 2x_1x_2 - 2|x_1x_2| \\ &= (k-2)^2 + k^2/2 - |k^2/2| = (k-2)^2. \end{aligned}$$

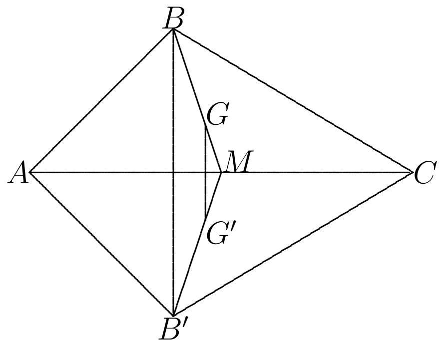
The solutions are $(x_1, x_2) = (-2, 0)$ and $(1+\sqrt{5}, 1-\sqrt{5})$.

19. 24 and 42. [24] n must have just two digits since $n \ge 100$ implies $n \cdot r(n) \ge 100n \ge 90n + 1000 > 10n + 1000 \ge p(n) + 1000$. Let $n = 10a + b$. Then $(10a + b)(10b + a) = ab + 1000$ so $10ab + a^2 + b^2 = 100$ or $(a-b)^2 + 12ab = 100$. One easily checks that $\{a, b\} = \{2, 4\}$ is the only solution.

20. 629. [9] The graphs of $y = x - \frac{\pi}{2}$ and $y = \tan x$ intersect exactly once in each interval $(\frac{(2k-1)\pi}{2}, \frac{(2k+1)}{2})$ for $k = 1, 2, \ldots$. The 400th solution will satisfy $\frac{799\pi}{2} < x < \frac{801\pi}{2}$. At this intersection, $y = x - \frac{\pi}{2}$ is very large, and so consideration of the graph of $y = \tan x$ shows that x will be only very slightly less than $801\pi/2$. So $\frac{x}{2}$ is very close to $200\pi + \frac{\pi}{4} = 628.32 + .78 = 629.1$.

21. 81. [8] The points of the form $(3\cos\alpha, 3\sin\alpha) + (5\sin\beta, 5\cos\beta)$ are those which are at distance 5 from a point on the circle of radius 3 centered at the origin. We wish to minimize the squared distance from any such points to the point $(8, 15)$. The closest such point will be at distance $3 + 5 = 8$ from the origin along the line from $(0, 0)$ to $(8, 15)$. Since the distance from $(0, 0)$ to $(8, 15)$ is 17, the minimal distance is $17 - 8 = 9$.

22. 48/17. [18] Let M be the midpoint of AC. By similar triangles, $GG' = BB'/3$. The area of triangle ABC equals $\frac{1}{4}BB' \cdot AC = \frac{17}{4}BB'$. On the other hand, by Heron's formula it equals $\sqrt{18(18-9)(18-10)(18-17)} = 36$. Thus $BB' = 144/17$.



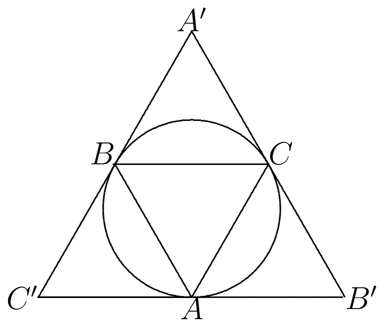
23. $[-1, 3+2\sqrt{5}]$. [4] Let $u = \log_a x$ and $v = \log_a y$. The constraints are $u^2 + v^2 - 2(u+v) \leq 2$ and $v \geq 1$. This is the upper half of the disk centered at $(1,1)$ with radius 2. We desire the possible values of $2u + v$ in this region. The smallest value will occur at the lower-left corner $(-1,1)$, where $2u+v = -1$. The largest will occur where a line of slope -2 is tangent to the semicircle. This point satisfies $v - 1 = \frac{1}{2}(u-1)$ and $(u-1)^2 + (v-1)^2 = 4$, hence is $(u,v) = (\frac{4}{\sqrt{5}} + 1, \frac{2}{\sqrt{5}} + 1)$, yielding $2u + v = 3 + 2\sqrt{5}$.

24. 99. [13] Since $\overline{def}_{11} \equiv d + e + f \bmod 5$, we want the number of triples (d,e,f) with entries in $\{0, \ldots, 10\}$ such that $d+e+f \equiv 0 \bmod 5$ and $0 < d < f$. For each (d,f) there are 2 or 3 choices for e, depending on whether $d + f \equiv 0 \bmod 5$. So our answer is twice the number of $0 < d < f \leq 10$ plus the number of choices with $d + f \equiv 0 \bmod 5$. The former is $\binom{10}{2}$, while the latter is 9: (1,4), (2,3), (1,9), (2,8), (3,7), (4,6), (5,10), (6,9), (7,8). Thus the answer is $2\binom{10}{2} + 9$.

25. $80/3$. [10] We have $2\angle C'AB = 180 - \angle C' = \widehat{AB} = 2\angle C$. Thus $C'A = 2/\cos(C'AB) = 2/\cos(C)$ and similarly $B'A = 3/\cos B$. By the Law of Cosines, $\cos(B) = 1/8$ and $\cos(C) = 3/4$. Therefore

$$B'C' = C'A + B'A = 2 \cdot \tfrac{4}{3} + 3 \cdot 8 = 80/3.$$

The figure below is not drawn to scale.



26. $3 + \frac{1}{2^{99}}$. [9] Since $g = f^{-1}$, the inverse of $g(2x+1)$ is $\frac{1}{2}(f(x)-1)$. This follows since the inverse of $(g\circ(+1)\circ(\cdot 2))$ is $((\cdot\frac{1}{2})\circ(-1)\circ f)$. Thus $\frac{1}{2}(f(x)-1)$ is the inverse of $G(x)$, so equals $f(x+1)-2$. So $f(x+1) = \frac{1}{2}f(x) + \frac{3}{2}$. Starting with $f(1) = 4$, we obtain successively $f(2) = 3 + \frac{1}{2}$, $f(3) = 3 + \frac{1}{4}$, and inductively $f(n) = 3 + \frac{1}{2^{n-1}}$.

27. 10. [11] Let a, b, and c denote the number of balls in bowls A, B, and C, respectively. Then $a + b + c = 27$ and $15a + 3b + 18c = 1 + \cdots + 27 = 378$ so $5a + b + 6c = 126$. Since $3b \geq 1 + \cdots + b = b(b+1)/2$, we must have $b \leq 5$, and so $b = 4$ or 5. Since $a + 5b = 6(a+b+c) - (5a+b+6c) = 36$, the two possibilities for (a, b, c) are $(11, 5, 11)$ and $(16, 4, 7)$.

In the first case, the largest possible number on the smallest-numbered ball in A is 10, when it has balls 10 through 20. In the second case, to have 16 balls averaging 15, there will certainly be some with numbers less than 10.

28. 15625. [3] Let P denote the desired answer. We have

$$\begin{aligned} 2^7 \cdot 5 \cdot P &= \prod_{i=1}^{7} 2r_i \prod_{1\le i<j\le 7} (r_i + r_j)^2 \\ &= \prod_{1\le i=j\le 7} (r_i + r_j) \prod_{1\le i<j\le 7} (r_i + r_j) \prod_{1\le j<i\le 7} (r_i + r_j) \\ &= \prod_{i=1}^{7}\prod_{j=1}^{7}(r_i + r_j). \end{aligned}$$

Note that $(x+r_1)\cdots(x+r_7) = x^7+5$. Thus for any i, $\prod_{j=1}^{7}(r_i + r_j) = r_i^7 + 5 = 10$. Therefore $2^7 \cdot 5 \cdot P = 10^7$, so $P = 5^6$.

29. 91125, 53125, and 95625. [2] Since $\overline{bcde} \mid \overline{abcde}$, it must divide $a \cdot 10^4 = a \cdot 2^4 \cdot 5^4$. Let $\overline{bcde} = x \cdot 2^k \cdot 5^\ell$, where $x \mid a$. We cannot have $\ell = 0$ since, if so, $\overline{bcde} \le 9 \cdot 2^4 = 144$ so $b = 0$. So $\ell > 0$ and then $k = 0$ for otherwise $e = 0$. If $\ell \le 2$, then $\overline{bcde} \le 9 \cdot 5^2 = 225$, so $b = 0$. Therefore $\ell = 3$ or 4.

Case 1: $\ell = 3$. Then $\overline{bcde} = x \cdot 125$. We must have $x = 9$ for otherwise $\overline{bcde}$ would contain a 0. Then $\overline{bcde} = 1125$ and $a = 9$, and 91125 works.

Case 2: $\ell = 4$. Then $\overline{bcde} = x \cdot 625$. Now x cannot be even or 1, since then $\overline{bcde}$ would have a 0. If $x = 3$, then $\overline{bcde} = 1875$, but 875 does not divide 1875. With $x = 5$, we get 53125, which works. If $x = 7$, then $\overline{bcde} = 4375$, but 375 is not a divisor of this. With $x = 9$, we get 95625, which works.

30. 24. [0] Any arrangement can be decomposed into disjoint cycles of length ≥ 1, where a *cycle* $(n_1, \ldots, n_k)$ means that card n_i is in slot n_{i+1} for $1 \le i < k$, and card n_k is in slot n_1. The goal is to have 20 length-1 cycles.

A cycle is *even* (resp. *odd*) if all its numbers are even (resp. odd), and otherwise is *mixed*. A transposition involving cards from different cycles is called a *fusion*, and otherwise is called a *break*. A fusion decreases the number of cycles by 1, while a break increases the number of cycles by 1. For example, if from cycle $(1, 2, 3, 4, 5)$ we exchange 1 and 4, the result is $(1, 5)(2, 3, 4)$, while if instead we exchanged 1 and 2, the result is $(2)(1, 3, 4, 5)$.

By fusing odd and even cycles no more than 5 times, we can make it so that all cycles of length > 1 are mixed. If $(\ldots, n, m, \ldots)$ is a cycle with n and m of opposite parity, interchanging them splits off a cycle (m) of length 1. At most 19 such moves gets us to where all cycles have length 1, as desired. So $5 + 19$ transpositions suffices.

We claim that an arrangement with all 10 evens in a single cycle and all odds split into 5 cycles of length 2 requires 24 transpositions. Five fusions are required to get the odds into mixed cycles. Now suppose that altogether F fusions and B breaks are performed to get to the goal of 20 length-1 cycles. Since we started with 6 cycles, we deduce that $20 = 6 + B - F$. Since $F \geq 5$, we obtain $B \geq 19$, so $F + B \geq 24$.