FROM REPRESENTATION THEORY TO HOMOTOPY GROUPS

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ABSTRACT. Bousfield recently gave an explicit formula for the odd-primary v_1-periodic homotopy groups of a finite H-space in terms of its K-theory and Adams operations. In this paper, we apply his result to give explicitly the v_1-periodic homotopy groups of E_8 localized at 3 and at 5, thus completing the determination of all odd-primary v_1-periodic homotopy groups of all compact simple Lie groups, a project suggested by Mimura in 1989.

The method involves no homotopy theoretic input and no spectral sequences. The input is the second exterior power operation in the representation ring of E_8, which we determine using specialized software. This can be interpreted as giving the Adams operation ψ^2 in $K(E_8)$.

Eigenvectors of ψ^2 must also be eigenvectors of ψ^k for any k. The determinant of the matrix of these eigenvectors is closely related to the homotopy decomposition of E_8 localized at each prime. By taking careful combinations of eigenvectors, we obtain a set of generators of $K(E_8)$ on which we have a nice formula for all Adams operations. Bousfield's theorem allows us to obtain from this the v_1-periodic homotopy groups.

1. INTRODUCTION

The p-primary v_1-periodic homotopy groups of a space X, denoted $v_1^{-1}\pi_*(X;p)$ or just $v_1^{-1}\pi_*(X)$, were defined in [17]. They are a localization of the actual homotopy groups, telling roughly the portion which is detected by K-theory and its operations. If X is a compact Lie group or spherically resolved space, each $v_1^{-1}\pi_i(X;p)$ is a direct summand of some actual homotopy group of X.

By use of a combination of homotopy-theoretic and unstable Novikov spectral sequence (UNSS) methods, the groups $v_1^{-1}\pi_*(X;p)$ were computed by the author and coworkers for the following compact simple Lie groups:

- X a classical group and p odd ([14]);
- X an exceptional Lie group with $H_*(X; \mathbf{Z})$ p-torsion-free ([10]);
- ($SU(n)$ or $Sp(n)$, 2) ([6],[9]);
- ($G_2, 2$) ([18]), (F_4 or E_6, 3) ([5]), and ($E_7, 3$) ([16]).

In [12], Bousfield takes a new approach to v_1-periodic homotopy groups. He shows that if X is a 1-connected finite H-space with $H_*(X; \mathbf{Q})$ associative, and p is an odd prime, then $v_1^{-1}\pi_*(X; p)$ can be obtained explicitly from $(K^*(X; \widehat{\mathbf{Z}}_p), \psi^p, \psi^r)$, where r is a generator of the group of units $(\mathbf{Z}/p^2)^\times$. We will review his result in Theorem 4.1.

Let X be any compact simple Lie group, and p an odd prime. In this paper, we will show how to compute the second exterior power λ^2 of generators of the representation ring $R(X)$, and use this to find a set of generators of $K^*(X; \widehat{\mathbf{Z}}_p)$ on which the Adams operations ψ^k behave in a nice way. From this, we use Bousfield's theorem to determine $v_1^{-1}\pi_*(X; p)$. This approach is totally algebraic. There is no homotopy theory (except that which went into proving Bousfield's theorem) and no UNSS.

We have used this approach to check the results obtained earlier by homotopy theory and the UNSS for (X, p) if $X = G_2$, F_4, E_6, or E_7, and $p \geq 3$, and $X = E_8$ and $p \geq 7$. All results are in agreement, except for one minor mistake in [5] in $v_1^{-1}\pi_*(F_4; 3)$, which will be discussed in Section 8. Also in Section 8 we will show how this new approach resolved two minor matters for $(E_7, 3)$ which had been left unresolved in [16].

In this paper, we will focus our attention on the calculation of $v_1^{-1}\pi_*(E_8; 5)$ and $v_1^{-1}\pi_*(E_8; 3)$, both of which are new. A main impediment toward finding $v_1^{-1}\pi_*(E_8; 5)$ had been uncertainty about a product decomposition which had been claimed by Harper in 1974 in [20, 4.4.1(b)]. In 1987, Kono questioned Harper's proof, and Harper agreed to Kono that his proof was flawed. Our methods show that indeed Harper's claim was incorrect; the asserted product decomposition does not exist. This will be explained more fully in Proposition 3.6.

Our main results are as follows, but we feel that the new methods introduced to obtain them are of much more interest than the results themselves. Let $\nu_p(-)$ denote the exponent of p in an integer.

Theorem 1.1. *Let $\langle m, k \rangle_r = \min(\nu_5(m-k)+r, k)$. Then*

$$v_1^{-1}\pi_{2m}(E_8; 5) \approx v_1^{-1}\pi_{2m-1}(E_8; 5)$$

$$\approx \begin{cases} 0 & \text{if } m \text{ is even} \\ \mathbf{Z}/5^{\max(\langle m,7 \rangle_4, \langle m,11 \rangle_4, \langle m,19 \rangle_4, \langle m,23 \rangle_4)} & \text{if } m \equiv 3 \mod 4. \end{cases}$$

If $m \equiv 1 \mod 4$, then $v_1^{-1}\pi_{2m}(E_8; 5) \approx \mathbf{Z}/5^e \oplus \mathbf{Z}/5$, and $v_1^{-1}\pi_{2m-1}(E_8; 5) \approx \mathbf{Z}/5^{e+1}$, where

$$e = \max(\langle m, 13 \rangle_2, \langle m, 17 \rangle_2, \langle m, 29 \rangle_2).$$

Theorem 1.2. *If $i \equiv 3, 4 \mod 4$, then $v_1^{-1}\pi_i(E_8; 3) = 0$. For any integer k,*

$$v_1^{-1}\pi_{4k+1}(E_8; 3) \approx v_1^{-1}\pi_{4k+2}(E_8; 3) \approx 3^e,$$

where

$$e = \begin{cases} \min(7 + \nu_3(k - 9 - 3^{13}), 21) & \text{if } k \equiv 0 \mod 9 \\ 6 & \text{if } k \equiv 1 \mod 3 \\ \min(9 + \nu_3(k - 11), 24) & \text{if } k \equiv 2 \mod 9 \\ \min(7 + \nu_3(k - 6 - 2 \cdot 3^7), 15) & \text{if } k \equiv 3, 6 \mod 9 \\ \min(10 + \nu_3(k - 14), 30) & \text{if } k \equiv 5 \mod 9 \\ \min(9 + \nu_3(k - 8), 18) & \text{if } k \equiv 8 \mod 9. \end{cases}$$

This completes the computation of $v_1^{-1}\pi_*(X; p)$ for all compact simple Lie groups X and odd primes p, a project which was suggested to the author by Mimura in 1989. The situation when $p = 2$, which was part of Mimura's suggested project, is much more delicate. In [7] and [8], Bendersky and the author have made some progress in adapting Bousfield's ideas to the prime 2.

There are many computations in this project which would be intractable to do by hand. Specialized software `LiE` ([27]) is used to determine the second exterior power operations in $R(E_8)$. A nontrivial algorithm was required to get this information into the form of an 8×8 matrix of integers, some of them 16 digits long, which can be interpreted as giving ψ^2 on a canonical basis of $PK^1(E_8)$, the primitive elements. This portion of the work will be described in Section 2.

The eigenvalues of this matrix are 2^e for $e \in R = \{1, 7, 11, 13, 17, 19, 23, 29\}$, corresponding to the fact that rationally E_8 is equivalent to $\prod_{e \in R} S^{2e+1}$. Using `Maple`, we find the associated eigenvectors. The determinant of the matrix of these eigenvectors

is

$$D = 2^{61}3^{32}5^{10}7^911^413^417^319^223^229.$$

This implies that localized at a prime p greater than 29, these eigenvectors span $PK^1(E_8)_{(p)}$, which then is isomorphic to $PK^1(\prod_R S^{2e+1})_{(p)}$ as a module over all Adams operations ψ^k, and hence by Bousfield's theorem has

$$v_1^{-1}\pi_*(E_8;p) \approx v_1^{-1}\pi_*(\prod_R S^{2e+1};p).$$

If, for example, $p = 29$, we can find two of the eight eigenvectors, v and w, for which $v' := (v - w)/29$ is integral. The set of vectors obtained from the eight eigenvectors by replacing v by v' has its determinant equal to $D/29$, which is a unit in $\mathbf{Z}_{(29)}$, and so this set spans $PK^1(E_8)_{(29)}$. The eigenvectors v and w correspond to eigenvalues 2^1 and 2^{29}. Then $\psi^k(v) = kv$ and $\psi^k(w) = k^{29}w$ for all integers k, and so we can determine $\psi^k(v')$, and from this use Bousfield's theorem to find $v_1^{-1}\pi_*(E_8;29)$, which agrees with that deduced in [10] from the decomposition

$$E_8 \simeq_{29} B(3,59) \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47}.$$

The point is that the value of the determinant D, computed blindly from representation theory, is intimately related to the decomposition of E_8 when localized at each prime. Note also that the matrix analysis shows that the portion of $K^1(E_8;\mathbf{Q})$ which must be modified to pass to $K^1(E_8;\mathbf{Z}_{(29)})$ is the portion related to S^3 and S^{59}, consistent with the homotopy analysis.

Because of the 5^{10} factor in D, we must 10 times replace vectors by $\frac{1}{5}$ times a difference of vectors in order to find a set of vectors whose determinant is a unit in $\mathbf{Z}_{(5)}$. On this set, an explicit formula for the Adams operations ψ^k can be given. This portion of the work will be described in Section 3. We also show there how performing these basis changes for all relevant primes enables a determination of the Adams operations ψ^k in $K^*(X)$ (not localized at a prime) for all exceptional Lie groups X and all k.

The Adams operation formulas are of the sort that allow us to draw inferences about attaching maps in the localized Lie groups. This new homotopy-theoretic information has been derived here just from our representation-based calculations together with Adams' e-invariant work ([3]). This will be discussed in Sections 5 and 6.

We feed this information into Bousfield's theorem, which, after a good deal of manipulation, yields the results for $v_1^{-1}\pi_*(E_8; 5)$. The computations for $v_1^{-1}\pi_{2m}(E_8)$ are given in Section 4, and those for $v_1^{-1}\pi_{2m-1}(E_8)$ are given in Section 5, which also includes some useful general results, such as periodicity of the number of summands, and the use of exact sequences. In Section 6, we perform a similar analysis to obtain the result in Theorem 1.2 for $v_1^{-1}\pi_*(E_8; 3)$.

Bendersky and Thompson ([11]) have recently constructed an unstable Bousfield-Kan spectral sequence based on K_*K, where K represents periodic K-theory. It possesses some advantages over the BP-based UNSS used in papers such as [10] and [16], especially in that $K_*(E_8)_{(p)}$ is a free commutative algebra, whereas $BP_*(E_8)$ is not. Using the result obtained in our Proposition 3.5 about the Adams operations in $K^*(E_8)_{(5)}$, which effectively implies that there is an α_3 attaching map between cells which would have been in separate factors if Harper's asserted splitting had been correct, together with their K_*K-based spectral sequence, Bendersky and Thompson are able to compute $v_1^{-1}\pi_*(E_8; 5)$ in a manner which is arguably more insightful than the computation here. Their method requires information about the homotopy theory of E_8 (e.g. Steenrod operations), while ours requires information about the representation theory. But so far homotopy theory has been unable to provide the complete picture (i.e., the α_3 attaching map), for which it had to rely on the representation-theoretic approach presented here. Moreover, our result for $v_1^{-1}\pi_*(E_8; 3)$ still seems totally inaccessible to UNSS-type methods.

2. REPRESENTATION THEORY AND ψ^2 IN K-THEORY

In this section, we use representation theory to determine the Adams operation ψ^2 in $K^*(E_8)$, and present an algorithm by which this can be done for any compact simple Lie group.

Let G be a simply-connected compact Lie group. Bousfield's approach to $v_1^{-1}\pi_*(G; p)$ ([12]) requires as input certain Adams operations on the primitives $PK^1(G; \widehat{\mathbf{Z}}_p)$. Bousfield suggested to the author the relationship with exterior powers in the representation ring $R(G)$ described in the next two paragraphs.

Let I denote the augmentation ideal in $R(G)$. Hodgkin's theorem ([21]) implies that there is an isomorphism

$$I/I^2 \to PK^{-1}(G), \qquad (2.1)$$

which may be viewed either as induced from the composition $R(G) \to K^0(BG, *) \to K^{-1}(G)$ or from Hodgkin's function β that views a representation $\rho : G \to U(n)$ as a homotopy class in $[G, U] = K^{-1}(G)$. Although Hodgkin doesn't write the isomorphism (2.1), he describes $R(G)$ in such a way that I/I^2 is clearly the free abelian group on the reduced basic representations $\widetilde{\rho}_1, \ldots, \widetilde{\rho}_l$, and shows that $PK^{-1}(G)$ is the free abelian group on $\beta(\widetilde{\rho}_1), \ldots, \beta(\widetilde{\rho}_l)$. The simple description of β makes it clear that (2.1) respects the exterior power operations λ^n.

Adams operations are related to exterior powers by the Newton formula

$$\psi^n(a) - \lambda^1(a)\psi^{n-1}(a) + \cdots + (-1)^{n-1}\lambda^{n-1}(a)\psi^1(a) + (-1)^n n\lambda^n(a) = 0,$$

which implies that $\psi^n = (-1)^{n+1}n\lambda^n$ in I/I^2. By [1, 5.3], ψ^n in $K^1(G)$ corresponds to ψ^n/n in $K^{-1}(G)$. Thus ψ^n in $PK^1(G)$ corresponds to $(-1)^{n+1}\lambda^n$ in I/I^2.

Now let G be a compact simple Lie group of rank l (e.g., E_8 of rank 8). The representation theory of G is equivalent to that of the associated Lie algebra $\mathcal{G}$. Associated to $\mathcal{G}$ is a set Λ of weights, a subset Λ^+ of dominant weights, and a subset $\{\lambda_1, \ldots, \lambda_l\} \subset \Lambda^+$ such that Λ (resp. Λ^+) is the free abelian group (resp. free abelian monoid) generated by $\lambda_1, \ldots, \lambda_l$. (e.g., [22, p.67].) The set Λ is given a partial order by $\sum m_i\lambda_i \leq \sum m'_i\lambda_i$ if and only if $m_i \leq m'_i$ for $1 \leq i \leq l$.

To each irreducible representation of $\mathcal{G}$ is associated a finite set of weights with multiplicities. It is a theorem that one of these weights is larger than all the others, and it occurs with multiplicity 1. (e.g., [22, §21.1].) This highest weight is dominant. The "highest weight" defines a function from the set of isomorphism classes of irreducible representations of $\mathcal{G}$ to Λ^+, and this function is bijective. It is a theorem (See, e.g., [21, 3.3]) that $R(\mathcal{G})$ is a polynomial algebra generated by the irreducible representations $\rho_1, \ldots, \rho_l$ which have $\lambda_1, \ldots, \lambda_l$ as highest weights.

If $\mathbf{m} = (m_1, \ldots, m_l)$ is an l-tuple of nonnegative integers, let $V(\mathbf{m})$ denote the unique irreducible representation with highest weight $m_1\lambda_1 + \cdots + m_l\lambda_l$. We will need three types of information about representations.

- The dimension of $V(\mathbf{m})$ (as a complex vector space).

- The second exterior power $\lambda^2(V(\mathbf{m}))$, expressed as $\sum c_j V(\mathbf{k}_j)$ for nonnegative integers c_j and l-tuples $\mathbf{k}_j$ of nonnegative integers.

- The tensor product $V(\mathbf{m}) \otimes V(\mathbf{n})$, expressed as $\sum c'_j V(\mathbf{k}'_j)$.

There are algorithms for each of these, implemented conveniently in the software LiE ([27]). For $\dim(V(\mathbf{m}))$, Weyl's formula ([22, p.139]) is used. For $\lambda^2(V(\mathbf{m}))$, a formula of [4] for symmetrized products is used. For $V(\mathbf{m}) \otimes V(\mathbf{n})$, Klimyk's formula ([23]) is used.

Let $\widetilde{\rho}_i = \rho_i - \dim(\rho_i)$. We need $\lambda^2(\widetilde{\rho}_i)$ as a linear combination of $\widetilde{\rho}_1, \dots, \widetilde{\rho}_l$ in I/I^2. The software gives us $\lambda^2(\rho_i)$ as $\sum c_j V(\mathbf{k}_j)$, and hence

$$
\begin{aligned}
\lambda^2(\widetilde{\rho}_i) &= \lambda^2(\rho_i) + \lambda^1(\rho_i)\lambda^1(-\dim(\rho_i)) + \lambda^2(-\dim(\rho_i)) \\
&= \sum c_j V(\mathbf{k}_j) - \dim(\rho_i)\rho_i + \binom{-\dim(\rho_i)}{2}. \qquad (2.2)
\end{aligned}
$$

We can iterate tensor product computations to write monomials $\rho_1^{e_1} \cdots \rho_l^{e_l}$ as linear combinations of $V(\mathbf{k})$'s, and then apply an easy row reduction to this result to write $V(\mathbf{k})$'s as polynomials in the ρ_i's, or, after manipulating polynomials, in the $\widetilde{\rho}_i$'s. In I/I^2, we ignore products of $\widetilde{\rho}_i$'s. Substituting the formulas for $V(\mathbf{k}_j)$ as linear combinations of $\widetilde{\rho}_i$'s into (2.2) yields the desired expression of $\lambda^2(\widetilde{\rho}_i)$ as a linear combination of $\widetilde{\rho}_j$'s in I/I^2.

We illustrate how this works in the simple example of the exceptional Lie group G_2, and then show how the computations can be expedited. The software tells us $\lambda^2(\rho_1) = \rho_2 + \rho_1$ and $\lambda^2(\rho_2) = V(3,0) + \rho_2$. (Remember, $\rho_2 = V(0,1)$.) Also, $\dim(\rho_1) = 7$ and $\dim(\rho_2) = 14$. Thus

$$\lambda^2(\widetilde{\rho}_1) = \rho_1 + \rho_2 - 7\rho_1 + \binom{-7}{2} = -6(\widetilde{\rho}_1 + 7) + (\widetilde{\rho}_2 + 14) + \tfrac{7 \cdot 8}{2} = -6\widetilde{\rho}_1 + \widetilde{\rho}_2.$$

To find $\lambda^2(\widetilde{\rho}_2)$, we need to express $V(3,0)$ as a polynomial in ρ_1 and ρ_2. The software tells us

$$
\begin{aligned}
\rho_1 \otimes \rho_1 &= V(2,0) + \rho_1 + \rho_2 + 1 \\
\rho_1 \otimes \rho_2 &= V(1,1) + V(2,0) + \rho_1 \\
\rho_1 \otimes V(2,0) &= V(3,0) + V(1,1) + V(2,0) + \rho_2 + \rho_1.
\end{aligned}
$$

This allows us to compute the second equation in

$$\lambda^2(\rho_2) = V(3,0) + \rho_2 = \rho_1^3 - 2\rho_1\rho_2 - \rho_1^2 - \rho_1,$$

from which we derive

$$
\begin{aligned}
\lambda^2(\widetilde{\rho}_2) &= (\rho_1^3 - 2\rho_1\rho_2 - \rho_1^2 - \rho_1) - 14\rho_2 + \binom{-14}{2} \\
&= \widetilde{\rho}_1^3 + 20\widetilde{\rho}_1^2 + 104\widetilde{\rho}_1 - 2\widetilde{\rho}_1\widetilde{\rho}_2 - 28\widetilde{\rho}_2 \\
&\equiv 104\widetilde{\rho}_1 - 28\widetilde{\rho}_2 \bmod I^2. \quad\quad\quad\quad (2.3)
\end{aligned}
$$

By our earlier remarks, this implies that $PK^1(G_2)$ has basis $\{x_1, x_2\}$ with $\psi^2(x_1) = 6x_1 - x_2$ and $\psi^2(x_2) = -104x_1 + 28x_2$.

This procedure can be expedited by just looking at linear terms. If f is an element of $R(G)$, let $L(f)$ denote the first-order terms when f is written as a polynomial in $\widetilde{\rho}_1, \ldots, \widetilde{\rho}_l$. We have

$$L(\lambda^2(\widetilde{\rho}_2)) = L(V(3,0) + \rho_2) - 14\widetilde{\rho}_2, \quad\quad\quad\quad (2.4)$$

and from the tensor product equations above, a type of differentiation yields

$$
\begin{aligned}
2\dim(\rho_1)\widetilde{\rho}_1 &= L(V(2,0)) + \widetilde{\rho}_1 + \widetilde{\rho}_2 \\
\dim(\rho_2)\widetilde{\rho}_1 + \dim(\rho_1)\widetilde{\rho}_2 &= L(V(1,1)) + L(V(2,0)) + \widetilde{\rho}_1 \\
\dim(V(2,0))\widetilde{\rho}_1 + \dim(\rho_1)L(V(2,0)) &= L(V(3,0)) + L(V(1,1)) \\
&\quad + L(V(2,0)) + \widetilde{\rho}_2 + \widetilde{\rho}_1.
\end{aligned}
$$

The first equation says $L(V(2,0)) = 13\widetilde{\rho}_1 - \widetilde{\rho}_2$, then the second says $L(V(1,1)) = 8\widetilde{\rho}_2$, and then, using $\dim(V(2,0)) = 27$, the third equation says $L(V(3,0)) = 104\widetilde{\rho}_1 - 15\widetilde{\rho}_2$, which when substituted into (2.4) yields (2.3).

The LiE program that implements this expedited algorithm for E_8 is listed and described in Section 7. The one subtlety is how to know which tensor products to compute. The dominant weights are ordered by height[1], which is the sum of the coefficients when they are written as roots. For example, in $\mathcal{G}_2$ the weights ρ_1 and ρ_2 correspond to roots $3\alpha_1 + 2\alpha_2$ and $2\alpha_1 + \alpha_2$, respectively, and so the height of $V(m_1, m_2)$ is $5m_1 + 3m_2$. An elementary result states that $V(\mathbf{m}) \otimes V(\mathbf{n}) = V(\mathbf{m+n}) +$

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[1]sometimes called "level"

terms of height less than that of $V(\mathbf{m}+\mathbf{n})$. For every term $V(\mathbf{l})$ with $\sum l_i > 1$ which occurs in $\lambda^2(\rho_i)$, we choose a way of writing $\mathbf{l} = \mathbf{m}+\mathbf{n}$, and differentiate the equation

$$V(\mathbf{m}) \otimes V(\mathbf{n}) = V(\mathbf{l}) + \text{terms of lower height}$$

to inductively obtain formulas for $L(V(\mathbf{l}))$. It can happen that $V(\mathbf{m}) \otimes V(\mathbf{n})$ might contain terms $V(\mathbf{k})$ which did not appear in $\lambda^2(\rho_i)$. If so, we also find $L(V(\mathbf{k}))$ by the same method, i.e., differentiating a formula for $V(\mathbf{a}) \otimes V(\mathbf{b})$ where $\mathbf{a}+\mathbf{b} = \mathbf{k}$.

When this algorithm is performed for E_8, we obtain the matrix (2.5) for λ^2 on the basis $\{\widetilde{\rho}_1, \ldots, \widetilde{\rho}_8\}$ of I/I^2. Thus, for example,

$$L(\lambda^2 \widetilde{\rho}_1) = -3628\widetilde{\rho}_1 - \widetilde{\rho}_2 + \widetilde{\rho}_3 + 3875\widetilde{\rho}_8.$$

$$\left(\begin{array}{cccc}
-3628 & 1829621 & 12625838007 & -1270362010619556 \\
-1 & -116621 & -146298269 & 18170270443687 \\
1 & -496 & -5835130 & 582917207249 \\
0 & 1 & 3875 & -468700376 \\
0 & 0 & -177629 & 26815340999 \\
0 & -247 & 12587859 & -2027479372896 \\
0 & 150877 & -392633383 & 78837408033778 \\
3875 & -4549375 & 10807381790 & -2860474034106800
\end{array}\right.$$

$$\left.\begin{array}{cccc}
2706011993074 & -401581533 & 0 & 0 \\
-40039592220 & 6661497 & 0 & 0 \\
-1242615998 & 185628 & 0 & 0 \\
1073250 & -249 & 0 & 0 \\
-68699627 & 27001 & -1 & 0 \\
5393300762 & -2538745 & 248 & 0 \\
-233942373952 & 156457497 & -30876 & 1 \\
9317251205935 & -7999393170 & 2573495 & -247
\end{array}\right) \qquad (2.5)$$

We obtain the following important corollary of this computation, where β is the isomorphism of (2.1).

Corollary 2.6. *With respect to the basis $\{\beta(\widetilde{\rho}_1), \ldots, \beta(\widetilde{\rho}_8)\}$ of $PK^1(E_8)$, the matrix of $-\psi^2$ is given by (2.5).*

3. NICE FORM FOR ψ^k IN $PK^1(E_8)_{(5)}$ AND $PK^1(X)$

If $p = 3$ or 5, then 2 generates $(\mathbf{Z}/p^2)^\times$, and so Bousfield's theorem requires knowledge of ψ^2 and ψ^p. A computation of ψ^p similar to that of the previous section could be made (provided enough computer time and space is available), and the results (Corollary 2.6 and its analogue) plugged into Bousfield's theorem to give results for

$v_1^{-1}\pi_*(E_8; 5)$. However, the matrix (2.5) is so unwieldy that it would be very difficult to obtain a nice form for the resulting groups. For this reason, we find a new basis of $PK^1(E_8)_{(5)}$ on which the action of ψ^2 has a nicer form. That is the purpose of this section. Moreover, as we shall see, the new basis will be one on which every ψ^k can be determined at the same time as ψ^2. We will also perform similar computations for Adams operations in $K(X)$ (unlocalized) for all exceptional Lie groups X.

To this end, we use `Maple` to find the eigenvalues and eigenvectors of a matrix M, which is defined to be the negative of the matrix (2.5). This is the matrix of ψ^2 on $PK^1(E_8)$. We are not surprised to find that the eigenvalues of M are 2^1, 2^7, 2^{11}, 2^{13}, 2^{17}, 2^{19}, 2^{23}, and 2^{29} because of the rational equivalence

$$E_8 \simeq_{\mathbf{Q}} S^3 \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47} \times S^{59} \tag{3.1}$$

and the fact that ψ^k acts as multiplication by k^n on $K^1(S^{2n+1})$.

A matrix whose columns are eigenvectors of M corresponding to the eigenvalues listed above in increasing order is

$$
\begin{pmatrix}
418105625 & 451155607289497 & -3133156733386433 & -2595116726135 \\
4168750 & 3797965233710 & 16166554278770 & 49280463350 \\
23125 & 20720212181 & 97338188051 & 212788925 \\
1 & 873857 & 5050967 & 7697 \\
377 & 326320702 & 2019676642 & 2670694 \\
119249 & 99339310201 & 777470688031 & 532675273 \\
27753998 & 16454843873197 & 398907853660267 & -470842549139 \\
9022308750 & -26386060414578330 & -79539474507550230 & 68079530157990
\end{pmatrix}
$$

$$
\begin{pmatrix}
-7044348025 & -114112039 & 17026841 & 2691065 \\
-3895749830 & -1864130 & -136850 & -38570 \\
21836035 & 56893 & -7787 & -1235 \\
1471 & 1 & 1 & 1 \\
635402 & 206 & 446 & -58 \\
294773639 & -86407 & -48007 & 4409 \\
-42760408957 & 12566861 & 3586541 & -174307 \\
3836612952570 & -810585690 & -191313690 & 6425670
\end{pmatrix} \tag{3.2}
$$

The numbers in the columns are coefficients with respect to the basis of Corollary 2.6.

The determinant of this matrix is a 68-digit integer which factors as

$$2^{61} 3^{32} 5^{10} 7^9 11^4 13^4 17^3 19^2 23^2 29 \tag{3.3}$$

This determinant gives a lot of information. First, it says that localized at a prime greater than 29, the eigenvectors form a basis, consistent with the known result that E_8 localized at such primes is equivalent to a product of spheres. Let

$$\{v_1, v_7, v_{11}, v_{13}, v_{17}, v_{19}, v_{23}, v_{29}\}$$

denote the columns of (3.2). It is very important to note that since these vectors satisfy $\psi^2(v_i) = 2^i v_i$, they correspond to the sphere factors in (3.1), and hence also satisfy

$$\psi^k(v_i) = k^i v_i \qquad (3.4)$$

for any positive integer k.

For primes p satisfying $11 \leq p \leq 29$, it was shown in [26] that E_8 is p-equivalent to a certain product of spheres and sphere bundles over spheres with α_1 attaching maps. For such primes, the number of sphere bundles equals the exponent of p in (3.3). We use $p = 23$ to illustrate how combinations of the eigenvectors in (3.2) correspond to these product decompositions, providing somewhat more detail than we did in our brief sketch for $p = 29$ in Section 1.

We note that $v_1' := (v_1 - v_{23})/23$ and $v_7' := (v_7 - 18v_{29})/23$ are integral. If v_1 and v_7 are replaced by v_1' and v_7' in (3.2), then the determinant is divided by 23^2, and so the new set of vectors is a basis for $PK^1(E_8)_{(23)}$. It follows from (3.4) that $\psi^k(v_1') = kv_1' + \frac{1}{23}(k - k^{23})v_{23}$ and $\psi^k(v_7') = k^7 v_7' + \frac{18}{23}(k^7 - k^{29})v_{29}$. This agrees with the determination of ψ^k in sphere bundles $B(3, 47)$ and $B(15, 59)$ with attaching maps a_1 given in [3]. Thus as Adams modules

$$PK^1(E_8)_{(23)} \approx PK^1(B(3, 47) \times B(15, 59) \times S^{23} \times S^{27} \times S^{35} \times S^{39})_{(23)}.$$

By Bousfield's theorem, these two spaces will have isomorphic v_1-periodic homotopy groups. Of course, we already knew that by the product decomposition of [26], but here we are getting it without relying on the [26] result. In [10], v_1-periodic homotopy groups of α_1 sphere bundles over spheres were determined by the UNSS, and $v_1^{-1}\pi_*(E_8; p)$ deduced for $p \geq 11$ using the product decomposition of [26]. In Proposition 5.5, we will determine the v_1-periodic homotopy groups of these sphere bundles by Bousfield's theorem, giving us a self-contained computation. This can be done for (E_8, p) for any prime $p \geq 11$.

When $p = 5$, we use `Maple` to help us find 10 combinations of vectors that are divisible by 5. These are

$$
\begin{aligned}
v_1' &= (v_1 + 2v_{13})/5 \\
v_{13}' &= (v_{13} + 3v_{17})/5 \\
v_{17}' &= (v_{17} - v_{29})/5 \\
v_{13}'' &= (v_{13}' + v_{17}' - v_{29})/5 \\
v_7' &= (v_7 - v_{11})/5 \\
v_{11}' &= (v_{11} - 2v_{19})/5 \\
v_{19}' &= (v_{19} - v_{23})/5 \\
v_7'' &= (v_7' - v_{11}' + 2v_{19}')/5 \\
v_{11}'' &= (v_{11}' + v_{19}' + 2v_{23})/5 \\
v_7''' &= (v_7'' - v_{19}' - 2v_{23})/5
\end{aligned}
$$

The way that these turn out to be grouped, with 1, 13, 17, and 29 related in one group, and 7, 11, 19, and 23 related in the other group, is consistent with Wilkerson's product decomposition ([31]) of $E_{8(5)}$ as a product of two spaces whose rational types correspond to these two groupings. The matrix $(v_1', v_{13}'', v_{17}', v_{29}, v_7''', v_{11}'', v_{19}', v_{23})$ has determinant a unit in $\mathbf{Z}_{(5)}$, and so its columns form a basis for $PK^1(E_8)_{(5)}$. We rename the classes $(x_1, x_{13}, x_{17}, x_{29}, x_7, x_{11}, x_{19}, x_{23})$ and compute $\psi^k(x_i)$ using (3.4). We obtain the following result.

Proposition 3.5. $PK^1(E_8)_{(5)}$ *has basis* $\{x_1, x_{13}, x_{17}, x_{29}, x_7, x_{11}, x_{19}, x_{23}\}$ *satisfying*

$$\psi^k(x_1) = k^1 x_1 + 10k(k^{12} - 1)x_{13} - 8k(k^{12} - 1)x_{17} + \tfrac{4}{5}k(k^{12} - 1)x_{29}$$

$$\psi^k(x_{13}) = k^{13}x_{13} + \tfrac{4}{5}k^{13}(k^4 - 1)x_{17} + \tfrac{2}{25}k^{13}(1 + 2k^4 - 3k^{16})x_{29}$$

$$\psi^k(x_{17}) = k^{17}x_{17} - \tfrac{1}{5}k^{17}(k^{12} - 1)x_{29}$$

$$\psi^k(x_{29}) = k^{29}x_{29}$$

$$\psi^k(x_7) = k^7 x_7 + \tfrac{2}{5}k^7(1 - k^4)x_{11} + \tfrac{1}{25}k^7(3 - 2k^4 - k^{12})x_{19}$$
$$\qquad\qquad + \tfrac{1}{125}k^7(32 + 16k^4 - k^{12} - 47k^{16})x_{23}$$

$$\psi^k(x_{11}) = k^{11}x_{11} + \tfrac{1}{5}k^{11}(1 - k^8)x_{19} + \tfrac{1}{25}k^{11}(9k^{12} - k^8 - 8)x_{23}$$

$$\psi^k(x_{19}) = k^{19}x_{19} + \tfrac{1}{5}k^{19}(1 - k^4)x_{23}$$

$$\psi^k(x_{23}) = k^{23}x_{23}$$

A formula $\psi^k(x_n) = k^n x_n + \tfrac{u}{5}k^n(k^{4m} - 1)x_{n+4m} + \cdots$, with u a unit in $\mathbf{Z}_{(5)}$ and $m \not\equiv 0 \bmod 5$, is of the type that would be obtained if the space had cells of dimension $2n + 1$ and $2n + 1 + 8m$ with attaching map α_m. The contributions of these cells to $v_1^{-1}\pi_*(X)$ will be as if they had the α_m attaching map. The formula for $\psi^k(x_{17})$ is, at the very least, strongly suggestive that in $E_{8(5)}$ the 35- and 59-cells are connected by α_3. This would contradict the product decomposition asserted in [20, 4.4.1b], which said that one Wilkerson factor could be further decomposed as $X(3, 59) \times X(27, 35)$. One could make the argument more precise either by noting that it is impossible that a space whose Adams operations can be written as in Proposition 3.5 can be decomposed in this way (i.e., no change of basis can split the Adams operations), or by noting that $v_1^{-1}\pi_*(E_8; 5)$, as computed in the next section, is incompatible with such a decomposition. Thus we obtain the following result.

Proposition 3.6. *In Wilkerson's decomposition ([31, 2.3]) of $(E_8)_{(5)}$ as $X_0 \times X_2$, both factors are indecomposable. In particular, the product decomposition of X_2 asserted in [20, 4.4.1b] is not valid.*

By performing, for all relevant primes, changes of basis of the sort illustrated above for E_8 when $p = 5$ or 23, we obtain, for each exceptional Lie group X, bases for $PK^1(X)$ on which we can compute ψ^k. We obtain the following results. In all of them, let $B_i = \beta(\widetilde{\rho}_i)$, where β and $\widetilde{\rho}_i$ are as in Section 2.

Proposition 3.7. *A basis for $PK^1(G_2)$ is given by $\{y_1, y_5\}$ with $y_1 = B_1$ and $y_5 = -4B_1 + B_2$. For all integers k,*

$$\begin{aligned} \psi^k(y_1) &= ky_1 + \tfrac{1}{30}(k - k^5)y_5 \\ \psi^k(y_5) &= k^5 y_5 \end{aligned}$$

The nice feature of this result and the subsequent ones for the other exceptional Lie groups is that they are a result about integral K-theory (i.e., not localized at a prime) and the Adams operations have a nice form (triangular, among other features). Since $PK^1(X)$ generates $K^*(X)$ for compact simple Lie groups X, multiplicativity of the Adams operations allows us to deduce the Adams operations on all of $K^*(X)$. Note that the classes y are subscripted by the exponent e such that $\psi^k(y) = k^e y +$ other terms.

The method of proof in each case is to

- use `LiE` to obtain the matrix of ψ^2 on $\{B_1, \ldots, B_l\}$ similarly to (2.5),

- use `Maple` to find eigenvectors of this matrix similarly to (3.2), and note that these vectors satisfy $\psi^k(v_i) = k^i v_i$ for all k,

- use `Maple` to repeatedly replace vectors v by $(v - w)/p$, where p is a prime which divides the determinant of the matrix of vectors and w is a linear combination of vectors which appear after v in the most recent set of vectors, until the determinant is ± 1, and

- use `Maple` and (3.4) to compute ψ^k on the final set of vectors, since they are explicit combinations of the eigenvectors.

In [28] and [29], Watanabe computed the Chern character on a certain set of generators of $K(G_2)$, $K(F_4)$, and $K(E_6)$, and in [30] he explained the well-known way in which this would allow one to determine the Adams operations. Using `LiE`, his set of generators can be expressed as linear combinations of ours, and the results for ψ^k which could be read off from his results for ch can be related to ours. We checked

this for G_2 and F_4 and found the results to be in agreement. However, it should be pointed out that his generators do not have the nice feature of having a triangular matrix for ψ^k. Also, in June 1998, Watanabe told the author that he felt that his methods would not work for E_7 and E_8. An internal check on our results, which was performed for each exceptional Lie group, is to transform the ψ^k formulas to the basis of B_i's, and then set $k = 2$ and compare with the formulas given by `LiE`.

Now we state the results for the other exceptional Lie groups.

Proposition 3.8. *A basis for $PK^1(F_4)$ is given by $\{y_1, y_5, y_7, y_{11}\}$ with $y_1 = -B_4$, $y_5 = B_1 - 3B_4$, $y_7 = -2B_1 + B_3 - 15B_4$, and $y_{11} = -6B_1 + B_2 - 11B_3 + 102B_4$. For all integers k,*

$$
\begin{aligned}
\psi^k(y_1) &= ky_1 - \tfrac{1}{10}(k - k^5)y_5 + (-\tfrac{1}{70}k + \tfrac{1}{120}k^5 + \tfrac{1}{168}k^7)y_7 \\
&\quad + (-\tfrac{1}{4620}k + \tfrac{1}{6720}k^5 + \tfrac{1}{13440}k^7 - \tfrac{1}{147840}k^{11})y_{11} \\
\psi^k(y_5) &= k^5 y_5 + \tfrac{1}{12}(k^5 - k^7)y_7 + (\tfrac{1}{672}k^5 - \tfrac{1}{960}k^7 - \tfrac{1}{2240}k^{11})y_{11} \\
\psi^k(y_7) &= k^7 y_7 + \tfrac{1}{80}(k^7 - k^{11})y_{11} \\
\psi^k(y_{11}) &= k^{11} y_{11}
\end{aligned}
$$

Proposition 3.9. *A basis for $PK^1(E_6)$ is given by $\{y_1, y_4, y_5, y_7, y_8, y_{11}\}$ with*

$$
\begin{aligned}
y_1 &= 2B_1 - B_2 + 3B_6 \\
y_4 &= -B_1 + B_6 \\
y_5 &= -2B_1 + B_2 - 2B_6 \\
y_7 &= -B_1 - 3B_2 + B_5 - 12B_6 \\
y_8 &= 11B_1 - B_3 + B_5 - 11B_6 \\
y_{11} &= 42B_1 - 21B_2 - 6B_3 + B_4 - 6B_5 + 42B_6.
\end{aligned}
$$

For all integers k,

$$\psi^k(y_1) = ky_1 - \tfrac{1}{2}(k - k^4)y_4 + \tfrac{11}{10}(k - k^5)y_5 + (\tfrac{1}{70}k - \tfrac{11}{120}k^5 + \tfrac{13}{168}k^7)y_7$$
$$+ (-\tfrac{1}{140}k + \tfrac{1}{480}k^4 + \tfrac{11}{240}k^5 - \tfrac{13}{336}k^7 - \tfrac{1}{480}k^8)y_8$$
$$+ (\tfrac{1}{4620}k - \tfrac{11}{6720}k^5 + \tfrac{13}{13440}k^7 + \tfrac{67}{147840}k^{11})y_{11}$$

$$\psi^k(y_4) = k^4 y_4 + \tfrac{1}{240}(k^4 - k^8)y_8$$

$$\psi^k(y_5) = k^5 y_5 + \tfrac{1}{12}(k^5 - k^7)y_7 - \tfrac{1}{24}(k^5 - k^7)y_8 + (\tfrac{1}{672}k^5 - \tfrac{1}{960}k^7 - \tfrac{1}{2240}k^{11})y_{11}$$

$$\psi^k(y_7) = k^7 y_7 - \tfrac{1}{2}(k^7 - k^8)y_8 + \tfrac{1}{80}(k^7 - k^{11})y_{11}$$

$$\psi^k(y_8) = k^8 y_8$$

$$\psi^k(y_{11}) = k^{11} y_{11}.$$

Proposition 3.10. *A basis for $PK^1(E_7)$ is given by $\{y_1, y_5, y_7, y_9, y_{11}, y_{13}, y_{17}\}$ with*

$$y_1 = 1873B_1 - 35B_2 + 15B_3 - 2B_5 + 287B_6 - 23056B_7$$

$$y_5 = -113B_1 - B_2 + 2B_3 - B_5 + 29B_6 - 547B_7$$

$$y_7 = 6B_1 - 6B_2 + 7B_6 - 216B_7$$

$$y_9 = 2B_1 - B_2 + B_6 - 30B_7$$

$$y_{11} = -184B_1 - B_2 + 3B_3 - B_5 + 21B_6 - 292B_7$$

$$y_{13} = 120B_1 - 5B_2 - 2B_3 + B_5 - 22B_6 + 328B_7$$

$$y_{17} = 1672B_1 - 252B_2 - 34B_3 + B_4 - 12B_5 + 177B_6 - 1344B_7.$$

For all integers k,

$$\psi^k(y_1) = ky_1 - \frac{22544}{5}(k - k^5)y_5 + \left(\frac{29367}{70}k - \frac{64814}{15}k^5 + \frac{3659}{42}k^7\right)y_7 + \left(\frac{35089}{21}k - \frac{57769}{45}k^5\right.$$
$$- \frac{36590}{63}k^7 + \frac{8714}{45}k^9\right)y_9 + \left(-\frac{25409}{2310}k + \frac{1409}{210}k^5 + \frac{3659}{1680}k^7 + \frac{39031}{18480}k^{11}\right)y_{11}$$
$$+ \left(-\frac{45223173}{10010}k + \frac{85358629}{18900}k^5 + \frac{464693}{181440}k^7 + \frac{4357}{5400}k^9 + \frac{507403}{221760}k^{11} - \frac{24684827}{5896800}k^{13}\right)y_{13}$$
$$+ \left(\frac{1}{1021020}k + \frac{2154361}{154791000}k^5 + \frac{3659}{684288}k^7 + \frac{4357}{1296000}k^9 + \frac{39031}{11176704}k^{11}\right.$$
$$- \frac{24684827}{707616000}k^{13} + \frac{8118218221}{926269344000}k^{17}\right)y_{17}$$

$$\psi^k(y_5) = k^5y_5 - \frac{23}{24}(k^5 - k^7)y_7 + \left(-\frac{41}{144}k^5 - \frac{115}{18}k^7 + \frac{961}{144}k^9\right)y_9 + \left(\frac{1}{672}k^5 + \frac{23}{960}k^7\right.$$
$$- \frac{57}{2240}k^{11}\right)y_{11} + \left(\frac{60581}{60480}k^5 + \frac{2921}{103680}k^7 + \frac{961}{34560}k^9 - \frac{247}{8960}k^{11} - \frac{106799}{103680}k^{13}\right)y_{13}$$
$$+ \left(\frac{1529}{495331200}k^5 + \frac{161}{2737152}k^7 + \frac{961}{8294400}k^9 - \frac{19}{451584}k^{11} - \frac{106799}{12441600}k^{13}\right.$$
$$+ \frac{1473021997}{174356582400}k^{17}\right)y_{17}$$

$$\psi^k(y_7) = k^7y_7 - \frac{20}{3}(k^7 - k^9)y_9 + \frac{1}{40}(k^7 - k^{11})y_{11} + \left(\frac{127}{4320}k^7 + \frac{1}{36}k^9 - \frac{13}{480}k^{11}\right.$$
$$- \frac{13}{432}k^{13}\right)y_{13} + \left(\frac{7}{114048}k^7 + \frac{1}{8640}k^9 - \frac{1}{24192}k^{11} - \frac{13}{51840}k^{13} + \frac{17}{147840}k^{17}\right)y_{17}$$

$$\psi^k(y_9) = k^9y_9 + \frac{1}{240}(k^9 - k^{13})y_{13} + \left(\frac{1}{57600}k^9 - \frac{1}{28800}k^{13} + \frac{1}{57600}k^{17}\right)y_{17}$$

$$\psi^k(y_{11}) = k^{11}y_{11} + \frac{13}{12}(k^{11} - k^{13})y_{13} + \left(\frac{5}{3024}k^{11} - \frac{13}{1440}k^{13} + \frac{223}{30240}k^{17}\right)y_{17}$$

$$\psi^k(y_{13}) = k^{13}y_{13} + \frac{1}{120}(k^{13} - k^{17})y_{17}$$

$$\psi^k(y_{17}) = k^{17}y_{17}$$

Before stating the final of these results about Adams operations in the K-theory of exceptional Lie groups, the case in which many of the numbers become ridiculously large, we point out two features. One is that the coefficient of each y_j in $\psi^k(y_i)$ is actually an integer, yielding integrality results. The other is that at least the second terms give information about attaching maps via primes occurring in denominators. This follows from [3], where it was shown that in a sphere bundle over a sphere with attaching map α_t with $t \not\equiv 0 \bmod p$ the Adams operations will be as described in our Proposition 5.5. Thus, for example, the 5s in the denominators of the second terms in the formulas for $\psi^k(y_1)$, $\psi^k(y_9)$, and $\psi^k(y_{13})$ in Proposition 3.10 are, at the very least, strongly suggestive that there are α_1 attaching maps from 1 to 5, 9 to 13, and 13 to 17 in E_7, and a similar deduction can be made at the prime 3. The same conclusion can be made from somewhat simpler formulas of $K(-)_{(p)}$, but here we are getting information about all primes at once.

Proposition 3.11. *A basis for* $PK^1(E_8)$ *is given by* $\{y_1, y_7, y_{11}, y_{13}, y_{17}, y_{19}, y_{23}, y_{29}\}$ *with*

$$
\begin{aligned}
y_1 &= -784157B_1 - 30713B_2 - 218B_3 + 13B_5 - 919B_6 + 950380B_7 - 153687494B_8 \\
y_7 &= 224745B_1 - 2221B_2 - 101B_3 + 9B_5 - 950B_6 + 69688B_7 - 3701825B_8 \\
y_{11} &= 100088B_1 - 982B_2 - 45B_3 + 4B_5 - 422B_6 + 30898B_7 - 1636120B_8 \\
y_{13} &= 100091B_1 - 982B_2 - 45B_3 + 4B_5 - 422B_6 + 30897B_7 - 1635950B_8 \\
y_{17} &= 71682B_1 - 788B_2 - 32B_3 + 3B_5 - 318B_6 + 23441B_7 - 1244530B_8 \\
y_{19} &= 26482B_1 - 223B_2 - 12B_3 + B_5 - 105B_6 + 7632B_7 - 403600B_8 \\
y_{23} &= 28444B_1 - 195B_2 - 13B_3 + B_5 - 104B_6 + 7462B_7 - 392340B_8 \\
y_{29} &= 2691065B_1 - 38570B_2 - 1235B_3 + B_4 - 58B_5 + 4409B_6 \\
&\quad - 174307B_7 + 6425670B_8.
\end{aligned}
$$

For all integers k,

$$\psi^k(y_1) = ky_1 - \frac{1454165}{42}(k - k^7)y_7 + \left(\frac{182723259}{154}k - \frac{650593421}{336}k^7 + \frac{395881345}{528}k^{11}\right)y_{11}$$

$$+\left(-\frac{185634674}{143}k + \frac{50198648299}{24192}k^7 - \frac{9105270935}{12672}k^{11} - \frac{9172441277}{157248}k^{13}\right)y_{13}$$

$$+\left(\frac{1116883611}{6188}k - \frac{332043745267}{1596672}k^7 - \frac{4354694795}{145152}k^{11} + \frac{541174035343}{9434880}k^{13}\right.$$

$$\left.+\frac{14942613523}{135717120}k^{17}\right)y_{17} + \left(\frac{10741728047}{149226}k - \frac{113260168654603}{1634592960}k^7 - \frac{326918814701}{38320128}k^{11}\right.$$

$$\left.+\frac{1201589807287}{182891520}k^{13} - \frac{1240236922409}{1628605440}k^{17} + \frac{2656402375049}{90348410880}k^{19}\right)y_{19}$$

$$+\left(\frac{293377022893}{2028117}k - \frac{2311774903640460613}{13338278553600}k^7 - \frac{6031690056666301}{228248616960}k^{11}\right.$$

$$+\frac{34318514761511237}{627683696640}k^{13} + \frac{354124997881577}{820817141760}k^{17} - \frac{634880167636711}{21683618611200}k^{19}$$

$$\left.+\frac{31696200305279479}{2454243253862400}k^{23}\right)y_{23} + \left(\frac{1}{38818159380}k + \frac{254146452881}{7362729761587200}k^7\right.$$

$$+\frac{399916721902123}{72128388948295680}k^{11} - \frac{70600280509069}{10243797929164800}k^{13} + \frac{21980584492333}{53779939128115200}k^{17}$$

$$\left.+\frac{2656402375049}{5724475313356800}k^{19} + \frac{31696200305279479}{1236938599946649600}k^{23} - \frac{205974908943845743817}{817863802284724715 5200}k^{27}\right)y_{27}$$

$$\psi^k(y_7) = k^7y_7 - \frac{2237}{40}(k^7 - k^{11})y_{11} + \left(\frac{172603}{2880}k^7 - \frac{51451}{960}k^{11} - \frac{1825}{288}k^{13}\right)y_{13}$$

$$+\left(-\frac{1141699}{190080}k^7 - \frac{270677}{120960}k^{11} + \frac{21535}{3456}k^{13} + \frac{535673}{266112}k^{17}\right)y_{17} + \left(-\frac{389433691}{194594400}k^7\right.$$

$$\left.-\frac{9236573}{14515200}k^{11} + \frac{621595}{870912}k^{13} - \frac{44460859}{3193344}k^{17} + \frac{8970785339}{566092800}k^{19}\right)y_{19}$$

$$+\left(-\frac{7948805340661}{1587890304000}k^7 - \frac{1874574277103}{951035904000}k^{11} + \frac{1365640565}{229920768}k^{13}\right.$$

$$\left.+\frac{12694914427}{1609445376}k^{17} - \frac{2144017696021}{135862272000}k^{19} + \frac{794109865426391}{88921857024000}k^{23}\right)y_{23}$$

$$+\left(\frac{873857}{876515447808000}k^7 + \frac{11299013179}{27321359450112000}k^{11} - \frac{561881}{750461386752}k^{13}\right.$$

$$+\frac{787974983}{105450861035520}k^{17} + \frac{8970785339}{358676639808000}k^{19}$$

$$\left.+\frac{794109865426391}{44816615940096000}k^{23} - \frac{168990789500875519}{94007333595945369 60}k^{29}\right)y_{29}$$

$$\psi^k(y_{11}) = k^{11}y_{11} - \frac{23}{24}(k^{11} - k^{13})y_{13} + \left(-\frac{121}{3024}k^{11} - \frac{1357}{1440}k^{13} + \frac{29707}{30240}k^{17}\right)y_{17}$$

$$+\left(-\frac{4129}{362880}k^{11} - \frac{39169}{362880}k^{13} - \frac{2465681}{362880}k^{17} + \frac{2508979}{362880}k^{19}\right)y_{19}$$

$$+\left(-\frac{837985819}{23775897600}k^{11} - \frac{86054063}{95800320}k^{13} + \frac{704026193}{182891520}k^{17} - \frac{599645981}{87091200}k^{19}\right.$$

$$\left.+\frac{129765607619}{32691859200}k^{23}\right)y_{23} + \left(\frac{5050967}{683033986252800}k^{11} + \frac{177031}{1563461222400}k^{13}\right.$$

$$\left.+\frac{43698997}{11983052390400}k^{17} + \frac{228089}{2090188800}k^{19} + \frac{129765607619}{16476697036800}k^{23} - \frac{371040821209573}{464463110651904 00}k^{29}\right)y_{29}$$

$$\psi^k(y_{13}) = k^{13}y_{13} - \frac{59}{60}(k^{13} - k^{17})y_{17} + \left(-\frac{1703}{15120}k^{13} - \frac{4897}{720}k^{17} + \frac{5227}{756}k^{19}\right)y_{19}$$

$$+\left(-\frac{3741481}{3991680}k^{13} + \frac{1398241}{362880}k^{17} - \frac{1249253}{181440}k^{19} + \frac{3961099}{997920}k^{23}\right)y_{23}$$

$$+\left(\frac{7697}{65144217600}k^{13} + \frac{86789}{23775897600}k^{17} + \frac{5227}{47900160}k^{19}\right.$$

$$\left.+\frac{3961099}{502951680}k^{23} - \frac{142072160653}{17784371404800}k^{29}\right)y_{29}$$

$$\psi^k(y_{17}) = k^{17}y_{17} - \frac{83}{12}(k^{17} - k^{19})y_{19} + \left(\frac{23699}{6048}k^{17} - \frac{19837}{2880}k^{19} + \frac{179587}{60480}k^{23}\right)y_{23}$$

$$+\left(\frac{1471}{396264960}k^{17} + \frac{83}{760320}k^{19} + \frac{179587}{30481920}k^{23} - \frac{26172961}{4358914560}k^{29}\right)y_{29}$$

$$\psi^k(y_{19}) = k^{19}y_{19} - \frac{239}{240}(k^{19} - k^{23}) + \left(\frac{1}{63360}k^{19} + \frac{239}{120960}k^{23} - \frac{265}{133056}k^{29}\right)y_{29}$$

$$\psi^k(y_{23}) = k^{23}y_{23} + \frac{1}{504}(k^{23} - k^{29})y_{29}$$

$$\psi^k(y_{29}) = k^{29}y_{29}$$

4. Determination of $v_1^{-1}\pi_{2m}(E_8; 5)$

Bousfield proved the following result in [12]. This is the result that has been called "Bousfield's theorem" throughout this paper.

Theorem 4.1. *Let X be a 1-connected finite H-space with $H_*(X; \mathbf{Q})$ associative, p an odd prime, and r a generator of $(\mathbf{Z}/p^2)^\times$. Then*

$$v_1^{-1}\pi_{2m}(X; p) = (\mathrm{coker}(\phi_m))^{\#} \ and \ v_1^{-1}\pi_{2m-1}(X; p) = (\ker(\phi_m))^{\#},$$

where $(-)^{\#}$ denotes the Pontryagin dual, and

$$\phi_m = \psi^r - r^m : PK^1(X; \widehat{\mathbf{Z}}_p)/\mathrm{im}(\psi^p) \to PK^1(X; \widehat{\mathbf{Z}}_p)/\mathrm{im}(\psi^p).$$

For the finite abelian groups with which we deal, $PK^1(X; \widehat{\mathbf{Z}}_p)/\mathrm{im}(\psi^p)$ is isomorphic to $PK^1(X; \mathbf{Z}_{(p)})/\mathrm{im}(\psi^p)$, and the only effect of the Pontryagin dual is to reverse the direction of arrows.

The following result is immediate from Proposition 3.5 and Theorem 4.1.

Proposition 4.2. $v_1^{-1}\pi_{2m}(E_8; 5) \approx A_m \oplus B_m$, *where*

$$A_m = \mathbf{Z}_{(5)}(x_1, x_{13}, x_{17}, x_{29})/(r_1, \dots, r_8)$$

and

$$B_m = \mathbf{Z}_{(5)}(x_7, x_{11}, x_{19}, x_{23})/(s_1, \dots, s_8),$$

where $\mathbf{Z}_{(5)}(-)$ denotes the free $\mathbf{Z}_{(5)}$-module on the indicated generators, and the relations are given as follows:

r_1 : $5^{29}x_{29}$;

r_2 : $5^{17}x_{17} - 5^{16}(5^{12} - 1)x_{29}$;

r_3 : $5^{13}x_{13} + 4 \cdot 5^{12}(5^4 - 1)x_{17} + 2 \cdot 5^{11}(1 + 2 \cdot 5^4 - 3 \cdot 5^{16})x_{29}$;

r_4 : $5x_1 + 10 \cdot 5(5^{12} - 1)x_{13} - 8 \cdot 5(5^{12} - 1)x_{17} + 4(5^{12} - 1)x_{29}$;

r_5 : $(2^{29} - 2^m)x_{29}$;

r_6 : $(2^{17} - 2^m)x_{17} - \frac{1}{5} \cdot 2^{17}(2^{12} - 1)x_{29}$;

r_7 : $(2^{13} - 2^m)x_{13} + 12 \cdot 2^{13}x_{17} + \frac{3}{25} \cdot 2^{14}(11 - 2^{16})x_{29}$;

r_8 : $(2 - 2^m)x_1 + 20(2^{12} - 1)x_{13} - 16(2^{12} - 1)x_{17} + \frac{8}{5}(2^{12} - 1)x_{29}$;

s_1 : $5^{23}x_{23}$;

s_2 : $5^{19}x_{19} + 5^{18}(1 - 5^4)x_{23}$;

s_3 : $5^{11}x_{11} + 5^{10}(1 - 5^8)x_{19} + 5^9(9 \cdot 5^{12} - 5^8 - 8)x_{23}$;

s_4 : $5^7x_7 + 2 \cdot 5^6(1 - 5^4)x_{11} + 5^5(3 - 2 \cdot 5^4 - 5^{12})x_{19}$

$\quad\quad +5^4(32 + 16 \cdot 5^4 - 5^{12} - 47 \cdot 5^{16})x_{23}$;

s_5 : $(2^{23} - 2^m)x_{23}$;

s_6 : $(2^{19} - 2^m)x_{19} - 3 \cdot 2^{19}x_{23}$;

s_7 : $(2^{11} - 2^m)x_{11} - 51 \cdot 2^{11}x_{19} + 183 \cdot 2^{14}x_{23}$;

s_8 : $(2^7 - 2^m)x_7 - 3 \cdot 2^8x_{11} - 165 \cdot 2^7x_{19} - 771 \cdot 2^{12}x_{23}$.

We shall analyze A_m first. We will make frequent use of the following well-known fact proved in [2, 2.12].

Proposition 4.3. *If r generates the group of units $(\mathbf{Z}/p^2)^\times$, then*

$$\nu_p(r^m - 1) = \begin{cases} 0 & \text{if } m \not\equiv 0 \mod p - 1 \\ 1 + \nu_p(m) & \text{if } m \equiv 0 \mod p - 1. \end{cases}$$

Relations r_5, r_6, r_7, and r_8 imply immediately that A_m is 0 if $m \not\equiv 1 \mod 4$. We will write $m = 4k + 1$ and divide r_5, r_6, r_7, and r_8 by the units 2^{29}, 2^{17}, 2^{13}, and 2,

respectively. We use r_6 to eliminate x_{29}, and replace it by $((1-2^{4k-16})/819)x_{17}$ in the other relations, and then use the modified r_7 to eliminate x_{17}, replacing it by a specific multiple of x_{13} in all relations. Next we add $\frac{1}{5}(2^{4k}-1)r_4$ to r_8 to eliminate x_1 from r_8. Now the modified r_4 is the only relation involving x_1, and it is of the form $5x_1 + \alpha 5^2 x_{13}$. This implies that $x_1 + 5\alpha x_{13}$ generates a $\mathbf{Z}/5$ direct summand. The remaining relations on x_{13} are as follows, obtained respectively from r_1 and r_5, r_2, r_3, and r_8.

t_1 : $5^{\min(28,\nu(k-7))+\nu(k-4)+\nu(k-3)+3}$

t_2 : $5^{16}(4095 + (5^{12}-1)(2^{4(k-4)}-1))$

t_3 : $5^{13}\Big(6\cdot 819 + 3\cdot 2621(2^{4k-16}-1)$

$$+\tfrac{1}{5}(2^{4k-12}-1)\big(2(5^4-1)819 - \tfrac{1}{5}(2^{4k-16}-1)(1+2\cdot 5^4 - 3\cdot 5^{16})\big)\Big)$$

t_4 : $5^2\Big(12\cdot 819 + 6\cdot 2621(2^{4k-16}-1) - \tfrac{1}{5}(2^{4k-12}-1)(4\cdot 819 + \tfrac{2}{5}(2^{4k-16}-1))$

$$+\tfrac{1}{5}(2^{4k}-1)(5^{12}-1)\Big(12 + \tfrac{6\cdot 2621}{819}(2^{4k-16}-1)$$

$$-\tfrac{2}{5}(2^{4k-12}-1)(2+\tfrac{1}{5}(2^{4k-16}-1)/819)\Big)\Big)$$

The relation t_4 can be manipulated to a unit times

$$\tfrac{1}{5}(2^{4k-16}-1)(2^{4k-28}-1)(2^{12}(2^{4k-12}-1) - 5^{12}(2^{4k}-1)),$$
$$(4.4)$$

while t_3 can be rewritten as

$$-2^{16}5^{11}(2^{4k-16}-1)(2^{4k-28}-1) - 2^{13}5^{15}(2^{4k-12}-1)(2^{4k-28}-1) + 3\cdot 5^{28}(2^{4k-16}-1),$$
$$(4.5)$$

and t_2 as

$$5^{28}(2^{4k-16}-1) - 5^{16}2^{12}(2^{4k-28}-1). \qquad (4.6)$$

These four relations on x_{13} result in a summand of order 5^e, where

- if $k \equiv 0, 1 \bmod 5$, then $e = 2$ from (4.4);
- if $k \equiv 3 \bmod 5$, then e is the minimum of $2 + \nu(k-3)$ from (4.4) or 13 from (4.5);

- if $k \equiv 4 \bmod 5$, then e is the minimum of $2 + \nu(k-4)$ from (4.4) or 17 from (4.6);
- if $k \equiv 2 \bmod 5$, then e is the minimum of $2 + \nu(k-7)$ from (4.4) or 29 from (4.5).

Thus A_m gives the groups claimed in Theorem 1.1 in the case $m \equiv 1 \bmod 4$.

Now we use the relations $s_1, \ldots, s_8$ to determine B_m. The analysis is similar to that just employed for A_m, but we use a different technique near the end. First we use $s_5, \ldots, s_8$ to see that $B_m = 0$ if $m \not\equiv 3 \bmod 4$. Now we write $m = 4k + 3$, and divide the last four relations by their initial 2-power. We use s_6 to eliminate x_{23}, and then s_7 to eliminate x_{19}. When these expressions are substituted into s_8, it becomes

$$(1 - 2^{4k-4})x_7 - ux_{11},$$

where

$$u = 6 + \big(165 + 257 \cdot 2^5(1 - 2^{4k-16})\big)(1 - 2^{4k-8})/(488 \cdot 2^{4k-16} - 437) \tag{4.7}$$

is a unit. Thus x_{11} can be eliminated, and so B_m is cyclic with generator x_7 and relations

w_1 : $5^{\min(26,4+\nu(k-5)+\nu(k-4)+\nu(k-2)+\nu(k-1))}$;

w_2 : $5^{20+\nu(k-2)+\nu(k-1)}(15 + (1 - 5^4)(1 - 2^{4k-16}))$;

w_3 : $5^{10+\nu(k-1)}\bigg(5^2(488 \cdot 2^{4k-16} - 437)$

$\qquad\qquad + (1 - 2^{4k-8})\big(5 - 5^9 + \tfrac{1}{3}(1 - 2^{4k-16})(9 \cdot 5^{12} - 5^8 - 8)\big)\bigg)$;

w_4 : $5^4\bigg(5^3 u + (1 - 2^{4k-4})\bigg(2(5^2 - 5^6) + \dfrac{(1 - 2^{4k-8})}{488 \cdot 2^{4k-16} - 437}$

$\qquad\qquad \cdot\big(15 - 2 \cdot 5^5 - 5^{13} + \tfrac{1}{3}(1 - 2^{4k-16})(32 + 16 \cdot 5^4 - 5^{12} - 47 \cdot 5^{16})\big)\bigg)\bigg),$

where u is as in (4.7). We easily read off from these that the order of B_m is $5^{\min(7,4+\nu(k-1))}$ if $k \equiv 1$ or $3 \bmod 5$.

Now let $P = 2^{4k}$, and write w_4 as a unit times 5^4 times the following expression.

$$5^3 \left(6 \cdot 2^{12}(488P - 437 \cdot 2^{16}) + 2^4 \left(165 \cdot 2^{16} + 257 \cdot 32(2^{16} - P)\right)(2^8 - P) \right)$$

$$+ (16 - P)\left(50 \cdot 2^8(1 - 5^4)(488P - 437 \cdot 2^{16}) + (2^8 - P)\left(2^{16}(15 - 2 \cdot 5^5 - 5^{13}) \right. \right.$$

$$\left. \left. + \tfrac{1}{3}(2^{16} - P)(32 + 16 \cdot 5^4 - 5^{12} - 47 \cdot 5^{16})\right) \right).$$

We use `Maple` to write this expression as $A + B \cdot P + C \cdot P^2 + D \cdot P^3$, where A, B, C, and D are certain explicit large integers. We note that for any positive integer e, this cubic expression can be rewritten as

$$a_0 + a_1(P - 2^e) + a_2(P - 2^e)^2 + a_3(P - 2^e)^3,$$

where

$$
\begin{aligned}
a_3 &= D \\
a_2 &= C + 3 \cdot 2^e D \\
a_1 &= B + 2^{e+1}C + 3 \cdot 2^{2e}D \\
a_0 &= A + 2^e B + 2^{2e}C + 2^{3e}D.
\end{aligned}
$$

With $e = 8$, `Maple` computes these a_i, from which we deduce that w_4 can be written as

$$5^4\left(u_1 5^7 + u_2 5^3(2^{4k} - 2^8) + u_3 5(2^{4k} - 2^8)^2 + u_4(2^{4k} - 2^8)^3\right),$$

where u_i are units in $\mathbf{Z}_{(5)}$. The relation given by this and w_1 when $k \equiv 2 \bmod 5$ is the desired $5^{\min(11, 4+\nu(k-2))}$, and w_2 and w_3 are easily seen to imply no additional restrictions.

If $k \equiv 4 \bmod 5$, we use $e = 16$ in the above method (with the same values of A, B, C, and D) and obtain the following form of w_4, where again u_i are units in $\mathbf{Z}_{(5)}$.

$$5^4\left(u_1 5^{15} + u_2 5^2(2^{4k} - 2^{16}) + u_3 5^2(2^{4k} - 2^{16})^2 + u_4(2^{4k} - 2^{16})^3\right)$$

This and w_1 give $5^{\min(19, 4+\nu(k-4))}$ as the relation. To see that w_3 can give no smaller relation, we can write it as

$$5^{10}\left((2^{4k-16} - 1)(u_5(2^{4k-16} - 1) + 5u_6) + 5^{10}u_7\right),$$

with u_i units.

If $k \equiv 5 \bmod 5$, we use $e = 20$ in the above method and find that w_4 can be written as

$$5^4(u_1 5^{19} + u_2 5^2(2^{4k} - 2^{20}) + u_3 5(2^{2k} - 2^{20})^2 + u_4(2^{2k} - 2^{20})^3).$$

This and w_1 give the desired $5^{\min(24, 4+\nu(k-5))}$. To see that w_2 cannot give a smaller relation in this case, we write it as a unit times

$$5^{20}(2^4(2^{4k-20} - 1) + 5^4(2^{4k-16} - 1)).$$

Finally, w_3 may be rewritten, using the method employed above for w_4, as

$$5^{24}u_0 + 5^{11}u_1(2^{2k} - 2^{20}) + 5^{10}u_2(2^{4k} - 2^{20})^2,$$

which implies that it cannot give a smaller relation here.

This completes the calculation of B_m, showing that it yields exactly the values of $v_1^{-1}\pi_{2m}(E_8; 5)$ claimed in Theorem 1.1 when $m \equiv 3 \bmod 4$. This completes the determination of $v_1^{-1}\pi_*(E_8; 5)$ when $*$ is even.

5. DETERMINATION OF $v_1^{-1}\pi_{2m-1}(E_8; 5)$

We could compute $v_1^{-1}\pi_{2m-1}(E_8; 5)$ using Theorem 4.1 by computing the relevant kernel. These computations are probably even more difficult than those of Section 4, and so, uncharacteristically, we follow a program of avoiding calculations as much as possible. A sequence of five propositions will yield the result for $v_1^{-1}\pi_{2m-1}(E_8; 5)$ claimed in Theorem 1.1.

The first proposition is elementary and well-known.

Proposition 5.1. *Let p be odd. If X is a simply-connected finite H-space with $H_*(X; \mathbf{Q})$ associative or X is a space built from a finite number of odd-dimensional spheres by fibrations, then $v_1^{-1}\pi_{2m-1}(X; p)$ and $v_1^{-1}\pi_{2m}(X; p)$ have the same order.*

Proof. For the first type of X, we use Theorem 4.1 and the observation that the kernel and cokernel of an endomorphism of a finite abelian group have the same order. For the second type of X, we use the argument employed for $SU(n)$ in [14, p.530]. ∎

Next we have the following recent observation of Bendersky.

Proposition 5.2. *If X is a space with an H-space exponent at p, the number of direct summands in $v_1^{-1}\pi_i(X;p)$ depends only on the residue mod q of i.*

Here we begin using $q = 2(p-1)$, and say that X has an H-space exponent at p if some p^e-power map on some iterated loop space of X is null-homotopic. Compact Lie groups and spheres have H-space exponents at all primes.

Proof of Proposition 5.2. There is a commutative diagram of isomorphisms

$$
\begin{array}{ccc}
\pi_i(X;\mathbf{Z}/p) & \approx & \pi_i(X) \otimes \mathbf{Z}/p \oplus \mathrm{Tor}(\pi_{i-1}(X),\mathbf{Z}/p) \\
\downarrow & & \downarrow \\
\pi_{i+kqp^e}(X;\mathbf{Z}/p) & \approx & \pi_{i+kqp^e}(X) \otimes \mathbf{Z}/p \oplus \mathrm{Tor}(\pi_{i-1+kqp^e}(X),\mathbf{Z}/p)
\end{array}
$$

where the vertical arrows are as described in [14, §2]. These arrows are defined using the null-homotopy of the p^e-power map. The one on the left can equivalently be defined using Adams maps of the mod p Moore space. The direct limit of these vertical morphisms defines the v_1-periodic homotopy groups. The diagram commutes because the morphisms $\pi_* X \to \pi_{*+kqp^e} X$ commute with multiplication by p.

Passing to the direct limit, we obtain an isomorphism

$$
v_1^{-1}\pi_i(X;\mathbf{Z}/p) \approx v_1^{-1}\pi_i(X) \otimes \mathbf{Z}/p \oplus \mathrm{Tor}(v_1^{-1}\pi_{i-1}(X),\mathbf{Z}/p).
$$

These will be finite $\mathbf{Z}_{(p)}$-modules, and if we denote by $\#(G)$ the number of direct summands in such a module, we obtain

$$
\#(v_1^{-1}\pi_i(X;\mathbf{Z}/p)) = \#(v_1^{-1}\pi_i(X)) + \#(v_1^{-1}\pi_{i-1}(X)).
$$

On the other hand, $v_1^{-1}\pi_i(X;\mathbf{Z}/p) \approx v_1^{-1}\pi_{i+q}(X;\mathbf{Z}/p)$ because the $v_1^{p^e}$-map which defines the direct limit can be chosen to be the composite of maps $\Sigma^q M(p) \to M(p)$. Thus $\#(v_1^{-1}\pi_i(X)) + \#(v_1^{-1}\pi_{i-1}(X))$ has period q. This implies that $\#(v_1^{-1}\pi_i(X))$ has period q. To see this, we note that if $\Delta = \#(v_1^{-1}\pi_q X) - \#(v_1^{-1}\pi_0 X)$, it follows that

$$
\#(v_1^{-1}\pi_{j+q}X) - \#(v_1^{-1}\pi_j X) = (-1)^j \Delta
$$

for all j, which implies that

$$
\#(v_1^{-1}\pi_{j+Lq}(X)) - \#(v_1^{-1}\pi_j(X)) = (-1)^j L\Delta.
$$

If $\Delta \neq 0$, let $j = 1$ if $\Delta > 0$ and $j = 0$ if $\Delta < 0$. This implies the absurd statement that $\#(v_1^{-1}\pi_{j+Lq}(X)) < 0$ for L sufficiently large, and so we conclude $\Delta = 0$. ∎

By these two propositions and the computation of Section 4, Theorem 1.1 will be proven once we show $v_1^{-1}\pi_{2m-1}(E_8; 5)$ is cyclic for some $m \equiv 1 \bmod 4$ and for some $m \equiv 3 \bmod 4$. This could be done by an explicit calculation, but we prefer to approach it by exact sequences. The following result is immediate from the Snake Lemma.

Proposition 5.3. *Suppose* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is a short exact sequence of p-adic Adams modules with* ψ^p *acting injectively on each. Let*

$$\phi_i = \psi^r - r^m : M_i / \operatorname{im}(\psi^p) \to M_i / \operatorname{im}(\psi^p).$$

Then there is an exact sequence

$$0 \to \ker \phi_1 \to \ker \phi_2 \to \ker \phi_3 \to \operatorname{coker} \phi_1 \to \operatorname{coker} \phi_2 \to \operatorname{coker} \phi_3 \to 0.$$

Theorem 4.1 and exactness of Pontryagin duality yield the following corollary.

Corollary 5.4. *Suppose maps* $X \to Y \to Z$ *induce a short exact sequence of Adams modules*

$$0 \to PK^1(Z; \widehat{\mathbf{Z}}_p) \to PK^1(Y; \widehat{\mathbf{Z}}_p) \to PK^1(X; \widehat{\mathbf{Z}}_p) \to 0,$$

with ψ^p *acting injectively on each. Then there is a long exact sequence*

$$\to v_1^{-1}\pi_{i+1}Z \to v_1^{-1}\pi_i X \to v_1^{-1}\pi_i Y \to v_1^{-1}\pi_i Z \to v_1^{-1}\pi_{i-1}X \to .$$

Next we have the following basic calculation, which is the Bousfield approach to a sphere bundle over sphere with attaching map α_t. This result is analogous to [10, 1.3], which was obtained using the UNSS.

Proposition 5.5. *If* $PK^1(X; \widehat{\mathbf{Z}}_p)$ *has generators* x *and* y *with* $\psi^k y = k^{n+t(p-1)}y$ *and* $\psi^k x = k^n x + \frac{u}{p}k^n(k^{t(p-1)} - 1)y$, *where* u *is a unit in* $\mathbf{Z}_{(p)}$, *and* $t \not\equiv 0 \bmod p$, *then the only nonzero groups* $v_1^{-1}\pi_*(X; p)$ *are*

$$v_1^{-1}\pi_{2n+qi}(X; p) \approx v_1^{-1}\pi_{2n+qi-1}(X; p) \approx \mathbf{Z}/p^e,$$

where

$$e = \begin{cases} \min(\nu_p(i) + 2, n) & \text{if } i \not\equiv t \ mod \ p \\ \min(\nu_p(i-t) + 2, t(p-1) + n) & \text{if } i \equiv t \ mod \ p, \ and \ n > 1 \\ \min(\nu_p(i - t - tp^{t(p-1)}) + 1, t(p-1) + 2) & \text{if } i \equiv t \ mod \ p, \ and \ n = 1 \end{cases}$$

Proof. We again make frequent use of Proposition 4.3. By Theorem 4.1, $v_1^{-1}\pi_{2m}(X; p)$ has generators x and y subject to relations

$$p^{n+t(p-1)}y$$

$$p^n x + up^{n-1}(p^{t(p-1)} - 1)y$$

$$(r^{n+t(p-1)} - r^m)y$$

$$(r^n - r^m)x + \tfrac{u}{p}r^n(r^{t(p-1)} - 1)y$$

The last two relations imply that the group is 0 unless $m \equiv n$ mod $p - 1$. We let $m = n + (p-1)i$. Since $t \not\equiv 0$ mod p, the fourth relation can be used to eliminate y. We obtain that the group is cyclic with generator x, and relations on x

$$p^{\nu(i)+1+\min(\nu(i-t)+1, n+t(p-1))}$$

$$(r^{t(p-1)} - 1)p^{n-1} - p^{n-1}(p^{t(p-1)} - 1)(1 - r^{(p-1)i}).$$

The second of these relations can be rewritten as

$$p^{n-1}\big(r^{t(p-1)}(1 - r^{(p-1)(i-t)}) + p^{t(p-1)}(r^{(p-1)t} - 1)\big).$$

Now inspection of the relations shows that if $i - t \not\equiv 0$ mod p, the first relation becomes $p^{\nu(i)+2}$, and the second relation becomes p^n. If $i - t \equiv 0$ mod p, then the first relation becomes $p^{\min(\nu(i-t)+2, n+t(p-1)+1)}$, while the second becomes $p^{n+\min(\nu(i-t), t(p-1))}$ unless $\nu(i - t) = t(p - 1)$. If $i - t = \alpha p^{t(p-1)}$ and we write $r^{p-1} = 1 + \beta p$ with $\beta \not\equiv 0$ mod p, then the second relation becomes $(i - \alpha)\beta p^{t(p-1)+1}$ mod $p^{t(p-1)+2}$. It is now routine to translate these relations when $i - t \equiv 0$ mod p into those claimed in the proposition. This establishes the results for the even-dimensional groups.

The odd groups could be computed directly; however, we can avoid computation as follows. By Proposition 5.1, the odd groups have the asserted order, and by Proposition 5.2 it suffices to show that any one of them is cyclic. By Proposition 5.3,

there is an exact sequence

$$0 \to v_1^{-1}\pi_{2m}S^{2n+1} \to v_1^{-1}\pi_{2m}X \to v_1^{-1}\pi_{2m}S^{2n+1+tq} \xrightarrow{\delta} v_1^{-1}\pi_{2m-1}S^{2n+1}$$
$$\to v_1^{-1}\pi_{2m-1}X \to v_1^{-1}\pi_{2m-1}S^{2n+1+tq} \to 0, \tag{5.6}$$

where $v_1^{-1}\pi_* S^{2m+1}$ are the groups associated to the Adams module $PK^1(S^{2m+1}; \widehat{\mathbf{Z}}_p)$, which is cyclic with $\psi^k = \cdot k^m$. It is easy to use Theorem 4.1 to compute that the only nonzero groups $v_1^{-1}\pi_* S^{2m+1}$ are

$$v_1^{-1}\pi_{2m+iq}S^{2m+1} \approx v_1^{-1}\pi_{2m+iq-1}S^{2m+1} \approx \mathbf{Z}/p^{\min(m,\nu(i)+1)}, \tag{5.7}$$

in agreement with well-known results. Choose $m = n + (p-1)(t + p^{n+t(p-1)})$. By (5.7) and our computation of $v_1^{-1}\pi_{\mathrm{ev}}X$, the exact sequence (5.6) becomes

$$0 \to \mathbf{Z}/p \to \mathbf{Z}/p^{n+(p-1)t} \to \mathbf{Z}/p^{n+(p-1)t} \xrightarrow{\delta} \mathbf{Z}/p$$
$$\to v_1^{-1}\pi_{2n+qt+qp^{n+t(p-1)}-1}X \to \mathbf{Z}/p^{n+(p-1)t} \to 0,$$

which implies that δ is surjective and $v_1^{-1}\pi_{2n+qt+qp^{n+t(p-1)}-1}X \approx \mathbf{Z}/p^{n+(p-1)t}$. $\blacksquare$

Now, the determination of $v_1^{-1}\pi_{\mathrm{odd}}(E_8; 5)$ is completed by using the observation which precedes Proposition 5.3 and the following result.

Proposition 5.8. *If $m = 4k+1$ and $k \equiv 0$ or $1 \mod 5$, then $v_1^{-1}\pi_{2m-1}(E_8; 5) \approx \mathbf{Z}/5^3$. If $m = 4k + 3$ and $k \equiv 3 \mod 5$, then $v_1^{-1}\pi_{2m-1}(E_8; 5) \approx \mathbf{Z}/5^4$.*

Proof. Let M_1^{29} (resp. M_7^{23}) denote the sub Adams module of $PK^1(E_8)_{(5)}$ generated by x_1, x_{13}, x_{17}, and x_{29} (resp. x_7, x_{11}, x_{19}, and x_{23}). (See Proposition 3.5.) If $i, j \in \{1, 13, 17, 29\}$ with $i \leq j$, let M_i^j denote the subquotient of M_1^{29} generated by those x_k in M_1^{29} with $i \leq k \leq j$, and similarly for subquotients of M_7^{23}. Let

$$\phi_i^j = \psi^2 - 2^m : M_i^j / \mathrm{im}(\psi^5) \to M_i^j / \mathrm{im}(\psi^5)$$

for a fixed value of m of the type specified in the proposition. Let $K_i^j = \ker(\phi_i^j)$ and $C_i^j = \mathrm{coker}(\phi_i^j)$.

We consider first the case $m = 4k + 1$ with $k \equiv 0$ or $1 \mod 5$. By (5.7) we have $K_i^i \approx C_i^i \approx \mathbf{Z}/5$ for $i \in \{1, 13, 17, 29\}$, and by Propositions 5.5 and 3.5,

$$K_{13}^{17} \approx C_{13}^{17} \approx K_{17}^{29} \approx C_{17}^{29} \approx \mathbf{Z}/5^2.$$

By Theorem 1.1, $C_1^{29} \approx \mathbf{Z}/5 \oplus \mathbf{Z}/5^2$, and, by a calculation similar to that of A_m in Section 4 but much easier, we obtain

$$C_{13}^{29} \approx \mathbf{Z}/5^3. \tag{5.9}$$

We will sketch this calculation at the end of this section. The results about C- and K-groups just listed imply, by just diagram chasing, that

$$K_1^{29} \approx \mathbf{Z}/5^3, \tag{5.10}$$

which implies the first part of the proposition by Theorem 4.1.

Before we present the simple proof of (5.10), we wish to present the motivation or underlying rationale for this argument and these results. It involves homotopy charts of the type used extensively in [16].

Diagram 5.11.
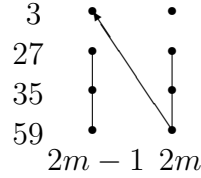
For the spheres S of dimensions 3, 27, 35, and 59 whose PK^1-Adams modules build this portion of $PK^1(E_8)_{(5)}$, and for the value of m being considered here,

$$v_1^{-1}\pi_{2m}(S) \approx v_1^{-1}\pi_{2m-1}(S) \approx \mathbf{Z}/5.$$

These are indicated by the dots. The vertical lines from 59 to 35, and from 35 to 27, are nontrivial multiplication by 5 obtained from Proposition 5.5, which applies because of the terms $-\frac{1}{5}k^{17}(k^{12}-1)x_{29}$ and $\frac{4}{5}k^{13}(k^4-1)x_{17}$ in Proposition 3.5. (The first of these is the all-important term which implies that Harper's product decomposition was false.) The computation in Section 4 of $A_m \approx \mathbf{Z}/5 \oplus \mathbf{Z}/5^2$ (for appropriate m) implies that there must be a differential from $v_1^{-1}\pi_{2m}(S^{59})$, but it wasn't clear whether it hit $v_1^{-1}\pi_{2m-1}(S^{27})$ or $v_1^{-1}\pi_{2m-1}(S^3)$. (Actually, the UNSS or the $\frac{4}{5}k(k^{12}-1)$-term in 3.5 make it pretty clear that it hits $v_1^{-1}\pi_{2m-1}(S^3)$.) Our computation that $C_{13}^{29} \approx \mathbf{Z}/5^3$ implies that there was no differential from 59 to 27, and so it must go from 59 to 3, as indicated in the diagram by the diagonal line.

But this is just one way of thinking. The result (5.10) can be obtained by diagram-chasing as follows. The claims made above about certain K-groups and C-groups

imply that the relevant exact sequences of 5.3 yield short exact sequences in the following commutative diagram, with all groups except the middle one given explicitly in the second diagram.

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{29}^{29} & \longrightarrow & K_{17}^{29} & \longrightarrow & K_{17}^{17} & \longrightarrow & 0 \\
& & \approx\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{29}^{29} & \longrightarrow & K_{13}^{29} & \longrightarrow & K_{13}^{17} & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & K_{13}^{13} & \xrightarrow{\approx} & K_{13}^{13} & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccccc}
\mathbf{Z}/5 & \longrightarrow & \mathbf{Z}/5^2 & \longrightarrow & \mathbf{Z}/5 \\
\approx\downarrow & & \downarrow & & \downarrow \\
\mathbf{Z}/5 & \longrightarrow & K_{13}^{29} & \longrightarrow & \mathbf{Z}/5^2 \\
& & \downarrow & & \downarrow \\
& & \mathbf{Z}/5 & \xrightarrow{\approx} & \mathbf{Z}/5
\end{array}
$$

It is easy to verify that in such a diagram, we must have $K_{13}^{29} \approx \mathbf{Z}/5^3$. Another exact sequence from 5.3 begins

$$0 \to K_{13}^{29} \to K_1^{29} \to .$$

Since $|K_1^{29}| = |C_1^{29}| = 5^3$, and K_1^{29} contains a cyclic subgroup, K_{13}^{29}, of order 5^3, we must have $K_1^{29} \approx \mathbf{Z}/5^3$, as claimed in (5.10).

The computation when $m = 4k + 3$ and $k \equiv 3 \bmod 5$ is easier. In this case, each of the four spheres yields a $\mathbf{Z}/5$, and each pair of consecutive spheres has a nontrivial $\cdot 5$ extension because of 5.5 and the terms $\frac{2}{5}k^7(1 - k^4)x_{11}$, $\frac{1}{5}k^{11}(1 - k^8)x_{19}$, and $\frac{1}{5}k^{19}(1 - k^4)x_{23}$ in 3.5. Since $|K_7^{23}| = |C_7^{23}| = 5^4$, we can deduce cyclicity of K_7^{23}

by chasing diagrams such as those used just above. The relevant homotopy chart is as below, with no differential.

$$
\begin{array}{c}
15 \\
23 \\
39 \\
47
\end{array}
\qquad
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\bullet & \bullet \\
\bullet & \bullet
\end{array}
$$
$$
2m-1 \quad 2m
$$

We close this section by sketching the proof of (5.9). We are computing A_m in Proposition 4.2, with x_1, r_4, and r_8 removed, and $m = 4k + 1$ with $k \equiv 0$ or 1 mod 5. In the analysis in Section 4, we will have generator x_{13} with relations t_1, t_2, and t_3. The relation t_1 says that $5^3 x_{13} = 0$, and the other relations involve much larger powers of 5. Thus the group is $\mathbf{Z}/5^3$, as claimed. ∎

6. Calculation of $v_1^{-1}\pi_*(E_8; 3)$

The computation of $v_1^{-1}\pi_*(E_8; 3)$ is performed similarly to that of $v_1^{-1}\pi_*(E_8; 5)$. One thing that makes the analysis more complicated is that all eight generators are related to one another by Adams operations, rather than being divided into two groups of four, as was the case for $(E_8, 5)$. It is more difficult to analyze an abelian group with 8 generators and 16 relations than to do it for two groups, each with 4 generators and 8 relations. We rely on Maple for every step of this computation.

For $i \in \{1, 7, 11, 13, 17, 19, 23, 29\}$, let v_i be the columns of (3.2), satisfying $\psi^k(v_i) = k^i v_i$. Then 32 times we replace vectors v by $v' := (v - w)/3$, with w a linear combination of vectors in the set which follow v, and v' still integral, and obtain finally a

basis $\{w_1, w_7, w_{11}, w_{13}, w_{17}, w_{19}, w_{23}, w_{29}\}$ for $PK^1(E_8; \mathbf{Z}_{(3)})$ defined by

$$
\begin{aligned}
w_1 &= v_1/3^2 + 730v_7/3^8 + 10v_{11}/3^{10} + 4v_{13}/3^8 + 28v_{17}/3^9 - 109v_{19}/3^8 \\
&\quad + 1364v_{23}/3^9 - 44078v_{29}/3^{10} \\
w_7 &= (v_7 + v_{11} + 7v_{13} + v_{17} + 92v_{19} - 14v_{23} - 994v_{29})/3^7 \\
w_{11} &= v_{11}/3^9 - v_{13}/3^6 - 2v_{17}/3^8 - 13v_{19}/3^6 + 83v_{23}/3^8 - 11885v_{29}/3^9 \\
w_{13} &= v_{13}/3^5 - v_{17}/3^6 + 7v_{19}/3^5 + 2v_{23}/3^6 + 227v_{29}/3^6 \\
w_{17} &= v_{17}/3^5 + v_{19}/3^3 + 4v_{23}/3^5 - 26v_{29}/3^5 \\
w_{19} &= v_{19}/3^2 - 2v_{23}/3^3 - v_{29}/3^3 \\
w_{23} &= (v_{23} - v_{29})/3^2 \\
w_{29} &= v_{29}.
\end{aligned}
$$

We chose to list here the basis in terms of eigenvectors v_i rather than the classes B_i, as was done in Propositions 3.7 to 3.11, primarily for the sake of variety.

By performing a matrix multiplication, we obtain the following formula for the Adams operations on the classes w_i. This is totally analogous to Theorem 3.11, except that here we are localized at 3.

Proposition 6.1. *On the basis $\{w_i\}$ of $PK^1(E_8; \mathbf{Z}_{(3)})$ just defined, the Adams operations are given, for any integer k, by*

$$\psi^k(w_1) = kw_1 - \frac{730}{3}(k - k^7)w_7 + (\frac{6560}{3}k - 2190k^7 + \frac{10}{3}k^{11})w_{11}$$

$$+(918k - \frac{24820}{27}k^7 + \frac{10}{9}k^{11} + \frac{4}{27}k^{13})w_{13}$$

$$+(\frac{1484}{3}k - \frac{40150}{81}k^7 + \frac{50}{81}k^{11} + \frac{4}{81}k^{13} + \frac{28}{81}k^{17})w_{17}$$

$$+(\frac{121}{3}k - \frac{29200}{729}k^7 + \frac{10}{243}k^{11} - \frac{40}{729}k^{13} - \frac{28}{243}k^{17} - \frac{109}{729}k^{19})w_{19}$$

$$+(-\frac{998}{3}k + \frac{727810}{2187}k^7 - \frac{1030}{2187}k^{11} - \frac{104}{2187}k^{13} - \frac{280}{2187}k^{17} - \frac{218}{2187}k^{19} + \frac{1364}{2187}k^{23})w_{23}$$

$$+(\frac{8479}{9}k - \frac{6187480}{6561}k^7 + \frac{99320}{59049}k^{11} - \frac{316}{6561}k^{13} + \frac{364}{19683}k^{17} - \frac{109}{6561}k^{19}$$

$$+\frac{1364}{19683}k^{23} - \frac{44078}{59049}k^{29})w_{29}$$

$$\psi^k(w_7) = k^7w_7 - 9(k^7 - k^{11})w_{11} + (-\frac{34}{9}k^7 + 3k^{11} + \frac{7}{9}k^{13})w_{13} + (-\frac{55}{27}k^7 + \frac{5}{3}k^{11}$$

$$+\frac{7}{27}k^{13} + \frac{1}{9}k^{17})w_{17} + (-\frac{40}{243}k^7 + \frac{1}{9}k^{11} - \frac{70}{243}k^{13} - \frac{1}{27}k^{17} + \frac{92}{243}k^{19})w_{19}$$

$$+(\frac{997}{729}k^7 - \frac{103}{81}k^{11} - \frac{182}{729}k^{13} - \frac{10}{243}k^{17} + \frac{184}{729}k^{19} - \frac{14}{243}k^{23})w_{23}$$

$$+(-\frac{8476}{2187}k^7 + \frac{9932}{2187}k^{11} - \frac{553}{2187}k^{13} + \frac{13}{2187}k^{17} + \frac{92}{2187}k^{19} - \frac{14}{2187}k^{23} - \frac{994}{2187}k^{29})w_{29}$$

$$\psi^k(w_{11}) = k^{11}w_{11} + \frac{1}{3}(k^{11} - k^{13})w_{13} + (\frac{5}{27}k^{11} - \frac{1}{9}k^{13} - \frac{2}{27}k^{17})w_{17}$$

$$+(\frac{1}{81}k^{11} + \frac{10}{81}k^{13} + \frac{2}{81}k^{17} - \frac{13}{81}k^{19})w_{19}$$

$$+(-\frac{103}{729}k^{11} + \frac{26}{243}k^{13} + \frac{20}{729}k^{17} - \frac{26}{243}k^{19} + \frac{83}{729}k^{23})w_{23}$$

$$+(\frac{9932}{19683}k^{11} + \frac{79}{729}k^{13} - \frac{26}{6561}k^{17} - \frac{13}{729}k^{19} + \frac{83}{6561}k^{23} - \frac{11885}{19683}k^{29})w_{29}$$

$$\psi^k(w_{13}) = k^{13}w_{13} + \frac{1}{3}(k^{13} - k^{17})w_{17} + (-\frac{10}{27}k^{13} + \frac{1}{9}k^{17} + \frac{7}{27}k^{19})w_{19}$$

$$+(-\frac{26}{81}k^{13} + \frac{10}{81}k^{17} + \frac{14}{81}k^{19} + \frac{2}{81}k^{23})w_{23}$$

$$+(-\frac{79}{243}k^{13} - \frac{13}{729}k^{17} + \frac{7}{243}k^{19} + \frac{2}{729}k^{23} + \frac{227}{729}k^{29})w_{29}$$

$$\psi^k(w_{17}) = k^{17}w_{17} - \frac{1}{3}(k^{17} - k^{19})w_{19} + (-\frac{10}{27}k^{17} + \frac{2}{9}k^{19} + \frac{4}{27}k^{23})w_{23}$$

$$+(\frac{13}{243}k^{17} + \frac{1}{27}k^{19} + \frac{4}{243}k^{23} - \frac{26}{243}k^{29})w_{29}$$

$$\psi^k(w_{19}) = k^{19}w_{19} + \frac{2}{3}(k^{19} - k^{23})w_{23} + (\frac{1}{9}k^{19} - \frac{2}{27}k^{23} - \frac{1}{27}k^{29})w_{29}$$

$$\psi^k(w_{23}) = k^{23}w_{23} + \frac{1}{9}(k^{23} - k^{29})w_{29}$$

$$\psi^k(w_{29}) = k^{29}w_{29}.$$

The second terms of the above expressions for $\psi^k(w_i)$ are, at the very least, strongly suggestive that ΦE_8 can be built by fibrations from ΦS^n's according to the scheme

in Diagram 6.2. Here Φ is Bousfield's functor ([12]) satisfying $\pi_*(\Phi X) \approx v_1^{-1}\pi_*(X)$ (localized at 3). We do not wish to belabor this here, since it is not necessary for our analysis. However, we note that it seems quite likely that this analysis could yield information about the attaching maps between cells of the 3-localizations of E_8 and ΩE_8, extending work in [25] and [19]. Our results here were previously inaccessible, since α_i is not detected by primary cohomology operations when $i > 1$.

Diagram 6.2.
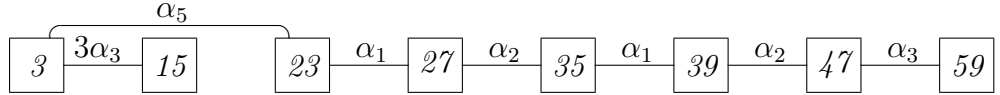
We return now to the application of Proposition 6.1 to determining $v_1^{-1}\pi_*(E_8; 3)$. By Theorem 4.1, $v_1^{-1}\pi_{2m}(E_8; 3)$ has a presentation with 8 generators and 16 relations, corresponding to $\psi^3(w_i)$ and $(\psi^2 - 2^m)(w_i)$, where we use 6.1 for ψ^2 and ψ^3. By Proposition 4.3, which we shall use frequently, the $\psi^2 - 2^m$ relations imply that $v_1^{-1}\pi_{2m}(E_8; 3) = 0$ if m is even, and so we let $m = 2k + 1$. We divide the $\psi^2 - 2^m$ relations by the unit 2, and arrange the relations as rows of a 16×8 matrix, with the ψ^3 relations listed first.

We use `Maple` to manipulate the matrix. The entry in position (15,8) is a unit, and so we pivot on it, and then delete the 15th row and the 8th column. This corresponds to writing the 8th generator as a combination of the other generators, then removing that generator and the relation which expressed it in terms of the others, while substituting this relation into all the others. In subsequent steps, we pivot on and then eliminate positions (14,7), (13,6), (12,5), (11,4), (1,3), and (8,2), ending up with a 9×1 matrix G. At each step, it is essential that the element on which we pivot is a unit. It was by no means clear at the outset that this could be done 7 times, for it has the important consequence that $v_1^{-1}\pi_{2m}(E_8; 3)$ is cyclic for each integer m.

In manipulating the matrix, we let, for $i \in \{0, 6, 10, 12, 16, 18, 22, 28\}$, $P_i = 2^{2k} - 2^i$. Thus, for example, the elements in position (9,1) and (10,2) of the initial matrix will be $-P_0$ and $-P_6$. After j steps of pivoting ($j \leq 7$), all entries in the matrix will be quotients of polynomials in the P_i of degree $\leq j + 1$. Since $P_i \equiv 0$ mod 3, mod 3 values of polynomials are determined by their constant term, which during the last

few steps will be up to 30 digits long. Up to this point, we have only cared about mod 3 values, to find units on which we could pivot, but in the next step we need more delicate information about exponents of 3.

We will have $v_1^{-1}\pi_{4k+2}(E_8; 3) \approx 3^e$, where e is the smallest exponent of 3 of the nine entries of the matrix G. The denominator polynomials are units, and can be ignored. `Maple` does some factoring automatically. For example, the fourth entry of G has a factor $3^{12}P_{10}P_{12}$, which we treat as $3^{14+\nu(k-5)+\nu(k-6)}$, since $\nu(P_{2j}) = 1 + \nu(k - j)$. Throughout this section, $\nu(-) = \nu_3(-)$.

Now we divide into cases depending upon the mod 9 value of k. We consider first the case $k \equiv 2 \equiv 11 \bmod 9$. After performing the preliminary simplifications described in the preceding paragraph, we replace each occurrence of P_{2j} by $R + 2^{22} - 2^{2j}$. Thus R is representing $2^{2k} - 2^{22}$, which satisfies $\nu(R) = 1 + \nu(k - 11)$. We find that the nine relations are, up to unit multiples of all terms, as follows. We emphasize that this assumes that $k \equiv 2 \bmod 9$, so that, for example, $\nu(k-8)$ has been replaced by 1, and $\nu(k - 10)$ by 0. We also point out that it seems to be infeasible to obtain these expressions by hand; their determination seems to require a computer.

$$3^{25} + 3^8R + 3^6R^2 + 3^4R^3 + 3^2R^4 + 3R^5 + R^6$$
$$3^2(3^{22} + 3^7R + 3^8R^2 + 3^3R^3 + 3^3R^4 + 3^2R^5 + R^6)$$
$$3^{10}(3^{16} + 3^6R + 3^5R^2 + 3^2R^3 + 3R^4 + R^5)$$
$$3^{15}(3^{12} + 3^4R + 3^2R^2 + 3R^3 + R^4)$$
$$3^{22}(3^7 + 3^3R + 3R^2 + R^3)$$
$$3^{27}(3^3 + 3R + R^2)$$
$$3^{37+\nu(k-11)}$$
$$3^{24} + 3^9R + 3^7R^2 + 3^5R^3 + 3^3R^4 + 3^2R^5 + 3R^6 + R^7$$
$$3^{10+\nu(k-11)}$$

Since $\nu(R) \geq 3$, we find that the term 3^8R in the first relation gives the smallest 3-power $(3^{9+\nu(k-11)})$ if $\nu(k - 11) \leq 15$, while the term 3^{24} in the second or eighth relation will be smallest if $\nu(k - 11) \geq 16$. This establishes the claim of Theorem 1.2 for $v_1^{-1}\pi_{2m}(E_8; 3)$ when $m = 2k + 1$ and $k \equiv 2 \bmod 9$.

The situation is similar when $k \equiv 8 \bmod 9$. If $R = 2^{2k} - 2^{16}$, then the first two relations are, up to unit multiples of all terms,

$$3^{19} + 3^8R + 3^6R^2 + 3^4R^3 + 3^2R^4 + 3R^5 + R^6$$

and

$$3^{18} + 3^9 R + 3^8 R^2 + 3^5 R^3 + 3^5 R^4 + 3^4 R^5 + 3^2 R^6,$$

while other relations involve larger exponents of 3. Note that these refer to precisely the same relations as the first two of the nine listed above; however, R now represents a different expression. Since $\nu(R) \geq 3$, the R^i-terms with $i \geq 2$ are more highly 3-divisible than the R^1-term, and so may be ignored. We find that the smallest exponent of 3 is $\min(9 + \nu(k-8), 18)$, with the $9 + \nu(k-8)$ coming from the first relation and the 18 from the second. This yields the case $k \equiv 8 \bmod 9$ for $v_1^{-1} \pi_{2m}(-)$ in Theorem 1.2. The case $k \equiv 5 \bmod 9$ follows similarly by noting that if $R = 2^{2k} - 2^{28}$, then the only significant terms are $3^{32} + 3^9 R + 3^6 R^2$ in the first relation and $3^{30} + 3^9 R$ in the second and eighth. If $\nu(R) = 3$, there was a possibility of cancellation of the R^1- and R^2-terms in the first relation, but not in the second.

The case $k \equiv 1 \bmod 3$ is easier. If $R = 2^{2k} - 2^8$, the first relation is $3^6 + 3^5 R + 3^4 R^2 + 3^3 R^3 + 3^2 R^4 + 3R^5 + R^6$, which gives a relation 3^6 if $\nu(R) \geq 2$, while the other relations are more highly 3-divisible.

The case $k \equiv 6 \bmod 9$ introduces a second-order effect. If $R = 2^{2k} - 2^{12}$, the only significant terms are $3^{14} + 3^6 R$ from the first relation and $3^{15} + 3^9 R$ from the second. The smallest 3-exponent is $7 + \nu(k-6)$ if $\nu(k-6) < 7$, and is 14 if $\nu(k-6) > 7$. If $\nu(k-6) = 7$, the two terms in the first relation are both 3^{14} times a unit, and we must analyze the mod 3 values of these units to tell whether the sum of these two terms is 3^{14} times a unit or is divisible by 3^{15}. As the second relation is 3^{15} times a unit, we need only evaluate the first relation mod 3^{15}.

`Maple` tells us that the unit coefficients of 3^{14} and $3^6 R$ in the first relation are both 1 mod 3. Thus mod 3^{15} the first relation is

$$3^{14} + 3^6((1+3)^k - (1+3)^6) \equiv 3^{14} + 3^6 \left(3(k-6) + 3^2 \left(\binom{k}{2} - \binom{6}{2}\right)\right).$$

If $k = 3^7 u$, this becomes $3^{14}(1 + u)$, and so is 3^{15} if $u \equiv 2 \bmod 3$, and 3^{14} if $u \equiv 1$. This yields the $\min(7 + \nu(k - 6 - 2 \cdot 3^7), 15)$ in 1.2 when $k \equiv 6 \bmod 9$. The case $k \equiv 0 \bmod 9$ follows similarly once `Maple` tells us that if $R = 2^{2k} - 2^{18}$, then the terms which can give smallest 3-power are $(3a+1)3^{20} + (3b+2)3^6 R$ from the first relation and $3^{21} + 3^9 R$ from the second.

The case $k \equiv 3$ mod 9 is easier. If $R = 2^{2k} - 2^6$, then the first relation begins $3^8 + 3^7 R + 3^4 R^2$, and other relations involve larger 3-powers. Thus $v_1^{-1} \pi_{2m}(-) \approx \mathbf{Z}/3^8$ in this case, which appears in 1.2 bundled along with the case $k \equiv 6$.

This concludes the computation of $v_1^{-1} \pi_{2m}(E_8; 3)$ in Theorem 1.2. The $v_1^{-1} \pi_{2m-1}(-)$ part of that theorem follows from Propositions 5.1 and 5.2 and the following result.

Proposition 6.3. *If $m = 3$, then $v_1^{-1} \pi_{2m-1}(E_8; 3) \approx \mathbf{Z}/3^6$.*

Proof. By Proposition 5.1 and the above computation that $v_1^{-1} \pi_6(E_8; 3) \approx \mathbf{Z}/3^6$, it suffices to show that $v_1^{-1} \pi_5(E_8; 3)$ has an element x satisfying $3^5 x \neq 0$. For $k = 2$ and 3, let M_k denote the transpose of the matrix of ψ^k with respect to the basis $\{w_1, w_7, w_{11}, w_{13}, w_{17}, w_{19}, w_{23}, w_{29}\}$ of Proposition 6.1. Thus M_k is the matrix whose rows have as entries the various coefficients in the equations of 6.1 with numerical values obtained by substituting 2 or 3 for k. Let $\Phi = M_2 - 8I$, an invertible matrix over $\mathbf{Z}_{(3)}$. Let $E = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$. Then EM_3 is the sum of the rows of M_3, and this combination of the w_i's is in $\mathrm{im}(\psi^3)$. Let $A = EM_3 \Phi^{-1}$. `Maple` computes

$$A = (\tfrac{-1}{2}, \tfrac{548957}{120}, \tfrac{-81625061}{2040}, \tfrac{-3845766033}{231880}, \tfrac{-605269537661}{759778008}, \tfrac{1212671517798439}{488157370140},$$
$$\tfrac{4278408406308902221}{200144521757400}, \tfrac{-2484645899025735277757029703}{2238578548469645972700}).$$

The second and third denominators are divisible by 3, but the others are not. Thus $3A$ represents an element of $PK^1(E_8; \mathbf{Z}_{(3)})$, and $3A\Phi$, which represents $\psi^2 - 2^3$ applied to this vector, is in $\mathrm{im}(\psi^3)$.

Thus $3A \in \ker(\psi^2 - 2^3 : PK^1(E_8; \mathbf{Z}_{(3)})/\mathrm{im}(\psi^3) \to PK^1(E_8; \mathbf{Z}_{(3)})/\mathrm{im}(\psi^3))$. We will be done once we have shown that $3^6 A \notin \mathrm{im}(\psi^3)$, for then $3^5(3A)$ is a nonzero element of the kernel, which, by Theorem 4.1, is isomorphic to $v_1^{-1} \pi_5(E_8; 3)$. This is verified by `Maple` by successively subtracting multiples of the rows of M_3 from $3^6 A$ to change the successive components of $3^6 A$ to 0. For the first 7 steps, the required multiplier will be in $\mathbf{Z}_{(3)}$, but when we get to the last step, our vector, which is equivalent to $3^6 A$ mod $\mathrm{im}(\psi^3)$, will be $(0, 0, 0, 0, 0, 0, 0, 3^{28} u)$, with u a unit in $\mathbf{Z}_{(3)}$. Since the corresponding relation in $\mathrm{im}(\psi^3)$ is 3^{29}, we conclude that $3^6 A \notin \mathrm{im}(\psi^3)$, as desired. ∎

This concludes the proof of Theorem 1.2, with the only input being λ^2 in $R(E_8)$. We find it useful to interpret our result in terms of homotopy charts such as Diagram

5.11. We emphasize that this analysis is not part of our proof, but rather is an attempt to understand how $v_1^{-1}\pi_*(E_8;3)$ is built from the v_1-periodic homotopy groups of the eight spheres which build it. In [16], in which the UNSS was the primary tool, these charts were an integral part of the argument, but here we just use our computation to see how those charts must have been filled in. We are not saying that E_8 is built from these eight spheres (It is not!), but rather that since $PK^1(E_8;\widehat{\mathbf{Z}}_p)$ as an Adams module is built from the related Adams modules for spheres (by our Theorem 3.5), $v_1^{-1}\pi_*(E_8;3)$ is built from that of the spheres, and we could probably conclude that Bousfield's ΦE_8 is built from Φ applied to the spheres.

An important ingredient in the charts such as 5.11 and those of [16] is the cyclicity of the groups for the "sphere bundles" described in Proposition 5.5. In $(E_8,3)$, we also encounter "sphere bundles" with α_p as attaching map, a case which was not considered in 5.5. Here we have the following result, whose proof we merely sketch.

Proposition 6.4. *If $PK^1(X;\widehat{\mathbf{Z}}_p)$ has generators x and y with $\psi^k y = k^{n+p(p-1)}y$ and $\psi^k x = k^n + \frac{u}{p^\epsilon}k^n(k^{p(p-1)} - 1)y$, where u is a unit in $\mathbf{Z}_{(p)}$ and $\epsilon = 1$ or 2, then the only nonzero groups $v_1^{-1}\pi_*(X;p)$ are:* [2]

a. If $\epsilon = 2$, then $v_1^{-1}\pi_{2n+qi}(X;p) \approx v_1^{-1}\pi_{2n+qi-1}(X;p) \approx \mathbf{Z}/p^e$, where

$$e = \begin{cases} \min(n-1,2) & \text{if } \nu(i) = 0 \\ \min(n,\nu(i)+3) & \text{if } \nu(i-p) = 1 \\ \min(n+p(p-1),\nu(i-p)+3) & \text{if } \nu(i-p) > 1 \text{ and } n \geq 4 \\ \min(4+p(p-1),\nu(i-p-p^{p(p-1)+1})+n-1) & \text{if } \nu(i-p) > 1 \text{ and } 2 \leq n \leq 3 \end{cases}$$

b. If $\epsilon = 1$ and $n \geq 2$, then $v_1^{-1}\pi_{2n+qi}(X;p) \approx v_1^{-1}\pi_{2n+qi-1}(X;p) \approx \mathbf{Z}/p \oplus \mathbf{Z}/p^e$, where

$$e = \begin{cases} 1 & \text{if } \nu(i) = 0 \\ \min(n,\nu(i)+2) & \text{if } \nu(i-p) = 1 \\ \min(n+p(p-1),\nu(i-p)+2) & \text{if } \nu(i-p) > 1 \text{ and } n \geq 3 \\ \min(3+p(p-1),\nu(i-p-p^{p(p-1)+1})+1) & \text{if } \nu(i-p) > 1 \text{ and } n = 2 \end{cases}$$

c. If $\epsilon = 1$ and $n = 1$, then

$$v_1^{-1}\pi_{2n+qi}(X;p) \approx v_1^{-1}\pi_{2n+qi-1}(X;p) \approx \mathbf{Z}/p^{\min(2+p(p-1),\nu(i-p)+1)}.$$

---

[2] If $\epsilon = 2$ and $n = 1$, then the coefficient of y in $\psi^p x$ is not in $\mathbf{Z}_{(3)}$, and so this case is excluded.

Proof. The proof follows closely that of Proposition 5.5 with $t = p$. We let $s = r^{p-1} = 1 + \beta p$, and note $\nu(s^i - 1) = 1 + \nu(i)$. We consider first the case $v_1^{-1}\pi_{2n+qi}(X; p)$.

First, let $\epsilon = 2$. As occurred in 5.5, the fourth relation allows one to eliminate y and deduce that the group is cyclic with relations (on generator x)

$$p^{1+\nu(i)}p^{\min(n+p(p-1),\nu(i-p)+1)} \quad \text{and} \quad p^{n-2}(p^{p(p-1)}(s^i - 1) + s^p - s^i).$$

The only case in which it is not easily seen that the minimal exponent is as claimed occurs when $n < 4$ and $i - p = u_0 p^{p(p-1)+1}$ with u_0 a unit in $\mathbf{Z}_{(p)}$. In this case, the first relation is $p^{p(p-1)+4}$, and so we analyze the second relation mod $p^{p(p-1)+4}$, obtaining

$$p^{n-2}\left(p^{p(p-1)}\big((1+\beta p)^i - 1\big) - (1+\beta p)^p\big((1+\beta p)^{i-p} - 1\big)\right)$$
$$\equiv \ \beta p^{n+p(p-1)}(1 - u_0)(1 - \beta p/2),$$

which yields the claim of the proposition.

Now let $\epsilon = 1$ and $n = 1$. We use the second of the four relations similar to those in the proof of 5.5 to eliminate y, again obtaining that $v_1^{-1}\pi_{2n+qi}(X; p)$ is cyclic with relations on generator x

$$p^{p(p-1)+2}, \ p(s^{i-p} - 1), \ \text{and} \ p^{p(p-1)}(s^i - 1) - s^p(s^{i-p} - 1),$$

whose minimal exponent of p is easily seen to be $\min(2 + p(p - 1), 1 + \nu(i - p))$, as claimed.

Finally, let $\epsilon = 1$ and $n > 1$. All terms in all four relations are divisible by p. We split off a $\mathbf{Z}/p$ generated by $1/p$ times the last relation, and replace uy by $(p(s^i - 1)/(s^p - 1))x$. The remaining summand has relations (on x)

$$p^{n+p(p-1)+\nu(i)}, \ p^{\nu(i)+\nu(i-p)+1}, \ \text{and} \ p^{n-2}(p^{p(p-1)}(s^i - 1) + s^p - s^i).$$

The only case in which it is not easily seen that the smallest exponent is as claimed occurs when $i - p = u_0 p^{p(p-1)+1}$ with u_0 a unit in $\mathbf{Z}_{(p)}$. Similarly to the case $\epsilon = 2$, the last relation becomes $\beta p^{2+p(p-1)}(1 - u_0)$ mod $p^{p(p-1)+3}$, which yields the claim of the proposition.

The result for $v_1^{-1}\pi_{2n+qi-1}(X; p)$ when $\epsilon = 2$ follows from the calculation of $v_{2n+qi}(X; p)$ just completed together with Propositions 5.1, 5.2, and 6.5.

Proposition 6.5. *If $\epsilon = 2$ and $\nu(i - p) \geq p(p - 1) + n$, then $v_1^{-1}\pi_{2n+qi-1}(X; p)$ is cyclic.*

Proof. The proof is similar to that of Proposition 5.5. In the case considered here, $|v_1^{-1}\pi_{2n+qi-1}(X)| = p^{p(p-1)+n}$ and, by (5.7), $v_1^{-1}\pi_{2n+qi-1}(S^{2n+pq+1}) \approx \mathbf{Z}/p^{p(p-1)+n}$, and so the result follows from the exact sequence

$$\to v_1^{-1}\pi_{2n+qi-1}(S^{2n+1}) \to v_1^{-1}\pi_{2n+qi-1}(X) \to v_1^{-1}\pi_{2n+qi-1}(S^{2n+pq+1}) \to 0.$$

∎

The result for $v_1^{-1}\pi_{2n+qi-1}(X)$ when $\epsilon = 1$ and $n = 1$ follows from Propositions 5.1 and 5.2 and the fact that if $i \not\equiv 0 \bmod p$, then $v_1^{-1}\pi_{2n+qi}(X)$ is cyclic, since it has order p.

The proof for $v_1^{-1}\pi_{2n+qi-1}(X)$ when $\epsilon = 1$ and $n \geq 2$ is more difficult, since it is not enough to verify it for one value of i. We focus on the case $\nu(i-p) = 1$, $\nu(i) \leq n-2$. Other cases are handled similarly. By Theorem 4.1

$$v_1^{-1}\pi_{2n+qi-1}(X) \approx \ker(\phi : G \to G),$$

where

$$
\begin{aligned}
G &= \langle x, y : \ p^{n+p(p-1)}y, \ p^n x + up^{n-1}(p^{p(p-1)} - 1)y\rangle \\
\phi(y) &= r^n(s^p - s^i)y \\
\phi(x) &= r^n((1 - s^i)x + \tfrac{u}{p}(s^p - 1)y).
\end{aligned}
$$

Using the second relation in G, we compute

$$\phi(p^{n-\nu(i)-1}x) = r^n u p^{n-\nu(i)-2}(s^p - s^i)(1 + Ap^{p(p-1)})y,$$

where $A := (s^i - 1)/(s^p - s^i)$ satisfies $\nu(A) = \nu(i) - 1$. Then

$$z := p^{n-\nu(i)-1}x - up^{n-\nu(i)-2}(1 + Ap^{p(p-1)})y$$

is in $\ker(\phi)$, and $p^{\nu(i)+1}z = -up^{n-1+p(p-1)}u'y$, where $u' = (s^p - 1)/(s^p - s^i)$ is a unit. Thus this element $p^{\nu(i)+1}z$ has order p in G, and so $\ker(\phi) \approx \mathbf{Z}/p^{\nu(i)+2} \oplus \mathbf{Z}/p$, with generators z and $uu'p^{n+p(p-1)-2}y + p^{\nu(i)}z$.   ∎

We list in Diagrams 6.6 and 6.7 some representative charts indicating how $v_1^{-1}\pi_{4k+\delta}(E_8)$, $\delta = 1, 2$, is built from the various $v_1^{-1}\pi_{4k+\delta}(S^{2n+1})$. These charts are of the type of Diagram 5.11 and those of [16]. For each of the four cases considered in each diagram, the left tower is building $v_1^{-1}\pi_{4k+1}(E_8)$ and the right tower is building $v_1^{-1}\pi_{4k+2}(E_8)$.

The numbers on the left side of each diagram are the dimensions of the spheres. Dots represent $\mathbf{Z}/3$, and a number e represents $\mathbf{Z}/p^e$. Vertical lines are extensions (multiplication by 3), as is the curved line from 23 to 3. Slanting arrows are boundary morphisms in exact sequences. In the third and fourth case of both diagrams, the boundary from the bottom generator is hitting a sum of two classes, which is relevant to the claim that the cokernel is cyclic.

The extensions from 59 to 47, and from 15 to 3, are consequences of Proposition 6.4 together with the terms $\frac{730}{3}(k-k^7)w_7$ and $\frac{1}{9}(k^{23}-k^{29})w_{29}$ in Proposition 6.1. The other extensions are derived similarly from Proposition 5.5. The most unexpected part of the charts is the extension in Diagram 6.7 from 27 into the sum of 23 and 15 in dimension $4k+2$. This was seen to be necessary in order that $v_1^{-1}\pi_{4k+2}(E_8)$ be cyclic in these cases. A check that this extension is really present was made by having `Maple` compute $v_1^{-1}\pi_{4k+2}(X(15,23,27))$, corresponding to the indicated subquotient Adams module from 6.1. This was done with $k=30$, corresponding to the first case in 6.7, and $\mathbf{Z}/3 \oplus \mathbf{Z}/3^6$ was obtained, consistent with the unexpected extension. This extension is probably also present in the cases of Diagram 6.6, but in those cases it would not affect the result.

There is also an unexpected extension in $4k+1$ from 23 to p times the generator in 15 when $k \equiv 6$ mod 9. This seemed necessary in order to get the correct answer when $\nu(k-6) \geq 12$, and was confirmed by a `Maple` computation that $v_1^{-1}\pi_{4k+2}(X(15,23,27)) \approx \mathbf{Z}/3 \oplus \mathbf{Z}/3^{\min(3+\nu(k-6),13)}$ if $k \equiv 6$ mod 9. The orders of the cyclic groups in the eight cases below are 6, $\nu(k-11)+9$, 24, 24, 8, 8, $\nu(k-6)+7$, and 14, in agreement with Theorem 1.2.
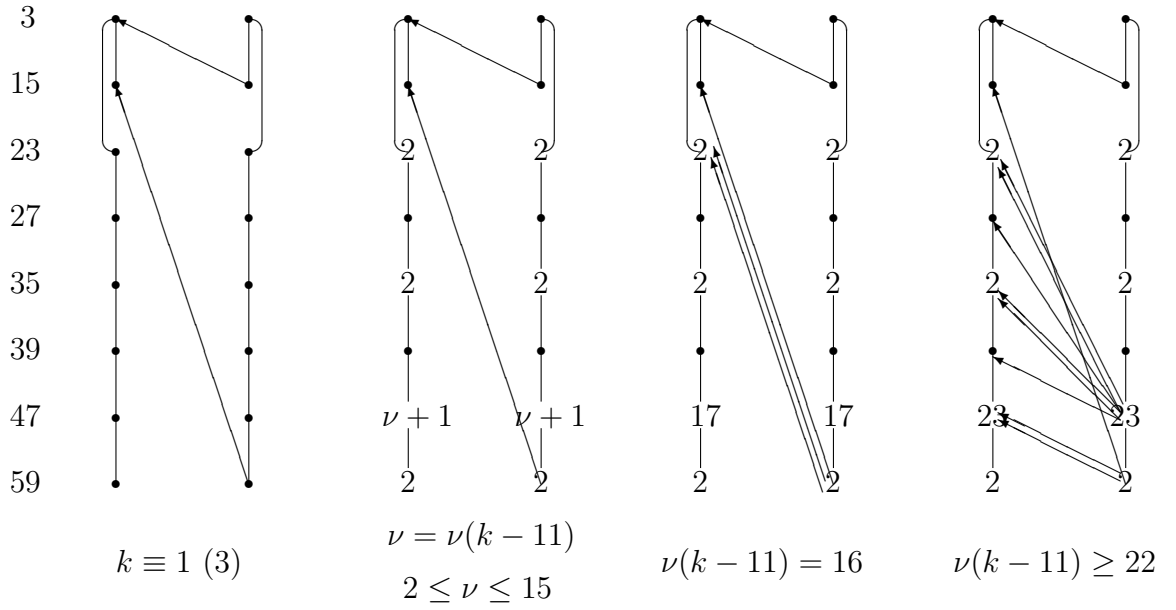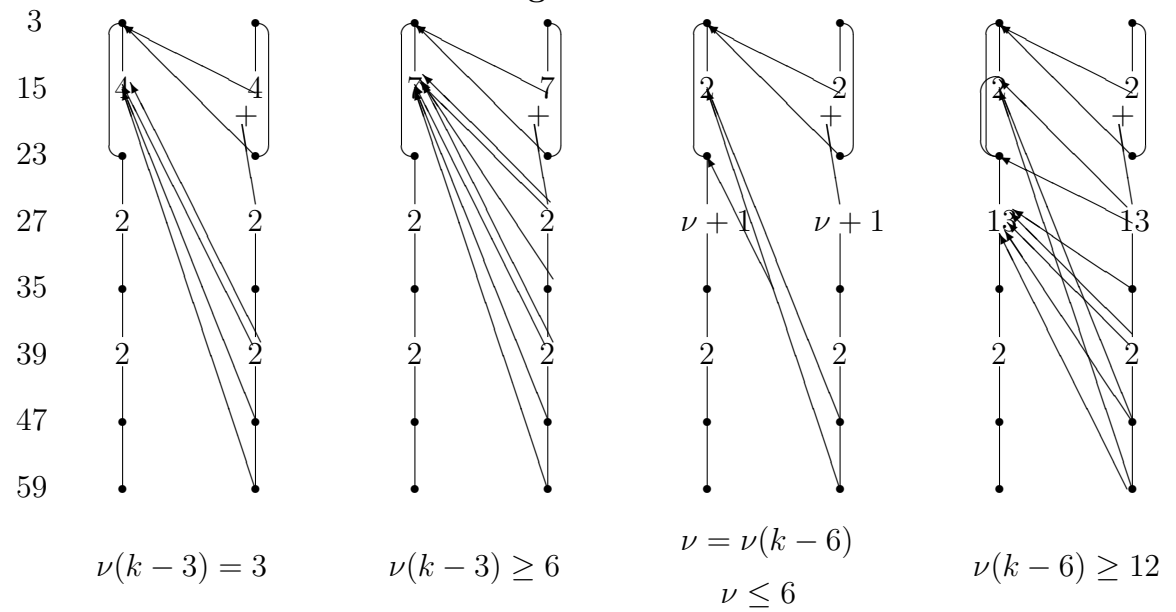
Diagram 6.6.

$$k \equiv 1 \ (3) \qquad \begin{array}{c} \nu = \nu(k-11) \\ 2 \leq \nu \leq 15 \end{array} \qquad \nu(k-11) = 16 \qquad \nu(k-11) \geq 22$$

Diagram 6.7.

$$\nu(k-3) = 3 \qquad \nu(k-3) \geq 6 \qquad \begin{array}{c} \nu = \nu(k-6) \\ \nu \leq 6 \end{array} \qquad \nu(k-6) \geq 12$$

The charts when $k \equiv 5,8 \bmod 9$ are very similar to those when $k \equiv 2$ given in Diagram 6.6, while those when $k \equiv 0 \bmod 9$ are very similar to those when $k \equiv 6$ given in Diagram 6.7. Again we emphasize that these charts are not a part of our

proof, but rather a way of interpreting our result in a way which is closer to past methods of calculating homotopy groups.

7. LiE PROGRAM FOR COMPUTING λ^2 IN $R(E_8)$

In this section we describe the program written in the specialized software LiE ([27]) to perform the calculation described in Section 2. We list the program and then describe what it is doing.

```
setdefault E8
on + height
mm=id(8)
for r row mm do
  ext=alt_tensor(2,r); extt=ext;
  p2=0X null(8); x=1;
  while x==1 do x=0;
    for i=1 to length(extt) do u=expon(extt,i);
      if u[1]+u[2]+u[3]+u[4]+u[5]+u[6]+u[7]+u[8]>1 then j=1;
        while u[j]==0 do j=j+1 od;
        v=null(8); v[j]=u[j]; w=u-v;
        if w==null(8) then w[j]=1; v[j]=v[j]-1 fi;
        p1=tensor(v,w); n=length(p1); utop=expon(p1,n);
        if extt | v==0 then x=1; p2=p2+1X v fi;
        if extt | w==0 then x=1; p2=p2+1X w fi;
        for k=1 to n-1 do a=expon(p1,k);
           if extt | a==0 then x=1; p2=p2+1X a fi od
      fi
    od;
    extt=extt+p2
  od;
  pdim=0X null(8); pder1=1X[1,0,0,0,0,0,0,0];
  pder2=1X[0,1,0,0,0,0,0,0]; pder3=1X[0,0,1,0,0,0,0,0];
  pder4=1X[0,0,0,1,0,0,0,0]; pder5=1X[0,0,0,0,1,0,0,0];
  pder6=1X[0,0,0,0,0,1,0,0]; pder7=1X[0,0,0,0,0,0,1,0];
```

```
pder8=1X[0,0,0,0,0,0,0,1];
for i=1 to length(extt) do
  u=expon(extt,i); pdim=pdim+dim(u)X u;
  if u[1]+u[2]+u[3]+u[4]+u[5]+u[6]+u[7]+u[8]>1 then j=1;
    while u[j]==0 do j=j+1 od;
    v=null(8); v[j]=u[j]; w=u-v;
    if w==null(8) then w[j]=1; v[j]=v[j]-1 fi;
    p1=tensor(v,w); n=length(p1);
    c1=dim(v)*(pder1 | w)+dim(w)*(pder1 | v);
    c2=dim(v)*(pder2 | w)+dim(w)*(pder2 | v);
    c3=dim(v)*(pder3 | w)+dim(w)*(pder3 | v);
    c4=dim(v)*(pder4 | w)+dim(w)*(pder4 | v);
    c5=dim(v)*(pder5 | w)+dim(w)*(pder5 | v);
    c6=dim(v)*(pder6 | w)+dim(w)*(pder6 | v);
    c7=dim(v)*(pder7 | w)+dim(w)*(pder7 | v);
    c8=dim(v)*(pder8 | w)+dim(w)*(pder8 | v);
    for i=1 to n-1 do c1=c1-coef(p1,i)*(pder1 | expon(p1,i));
      c2=c2-coef(p1,i)*(pder2 | expon(p1,i));
      c3=c3-coef(p1,i)*(pder3 | expon(p1,i));
      c4=c4-coef(p1,i)*(pder4 | expon(p1,i));
      c5=c5-coef(p1,i)*(pder5 | expon(p1,i));
      c6=c6-coef(p1,i)*(pder6 | expon(p1,i));
      c7=c7-coef(p1,i)*(pder7 | expon(p1,i));
      c8=c8-coef(p1,i)*(pder8 | expon(p1,i)) od;
    pder1=pder1+c1 X u; pder2=pder2+c2 X u;
    pder3=pder3+c3 X u; pder4=pder4+c4 X u;
    pder5=pder5+c5 X u; pder6=pder6+c6 X u;
    pder7=pder7+c7 X u; pder8=pder8+c8 X u
  fi
od;
der1=0; der2=0; der3=0; der4=0; der5=0; der6=0; der7=0; der8=0;
for i=1 to length(ext) do
```

```
     der1=der1+coef(ext,i)*(pder1 | expon(ext,i));
     der2=der2+coef(ext,i)*(pder2 | expon(ext,i));
     der3=der3+coef(ext,i)*(pder3 | expon(ext,i));
     der4=der4+coef(ext,i)*(pder4 | expon(ext,i));
     der5=der5+coef(ext,i)*(pder5 | expon(ext,i));
     der6=der6+coef(ext,i)*(pder6 | expon(ext,i));
     der7=der7+coef(ext,i)*(pder7 | expon(ext,i));
     der8=der8+coef(ext,i)*(pder8 | expon(ext,i))
  od;
  print(der1);print(der2);print(der3);print(der4);
  print(der5);print(der6);print(der7);print(der8)
od
```

The first line says that the program is always working with the Lie algebra E_8. The second line says that monomials are ordered by increasing height, a notion which was defined in Section 2. The first `for` loop, which extends throughout the program, lets `r` run over rows of an 8×8 identity matrix, with the ith iteration corresponding to computing $\lambda^2(\widetilde{\rho}_i)$. The variable `ext` is $\lambda^2(\rho_i)$ written as a polynomial whose exponents are dominant weights, and coefficients are their multiplicity. The variable `extt` will be a modified version of `ext`, expanded to include additional terms whose derivatives must be computed, as described in Section 2. The variable `x` tells whether any new terms were adjoined to `extt` in the most recent iteration. Each exponent `u` in `extt` with sum of entries greater than 1 is decomposed as `v+w`. The polynomial `p1` represents $V(\mathbf{v}) \otimes V(\mathbf{w})$. If `v` or `w` or any term of `p1` does not appear in `extt`, then it is adjoined to `extt`, because we will need to know its derivatives. The purpose of the 14-line `while` loop is just to adjoin these terms.

The variable `pdim` is a polynomial whose coefficient of `X u` is $\dim(V(\mathbf{u}))$, while variables `pder`j are polynomials whose coefficient of `X u` is $\partial_j(V(\mathbf{u}))$, the coefficient of $\widetilde{\rho}_j$ in $L(V(\mathbf{u}))$. Here `u` ranges over all terms in `extt`; these will depend upon which $\lambda^2(\widetilde{\rho}_i)$ we are computing. The long `for` loop computes these inductively. It still works with `extt`, which is $\lambda^2(\widetilde{\rho}_i)$ modified to include extra terms whose derivatives are needed in the induction. The variable `c`j is $\partial_j(V(\mathbf{u}))$, i.e., the coefficient of $\widetilde{\rho}_j$ in

$V(\mathbf{u})$. It is determined by writing $\mathbf{u} = \mathbf{v} + \mathbf{w}$, expanding

$$V(\mathbf{v}) \otimes V(\mathbf{w}) = V(\mathbf{u}) + \sum k_l V(\mathbf{t}_l)$$

with height$(\mathbf{t}_l) <$ height$(\mathbf{u})$, so that $\partial_j(V(\mathbf{t}_l))$ has already been computed, and computing

$$\partial_j(V(\mathbf{u})) = \dim(V(\mathbf{v}))\partial_j(V(\mathbf{w})) + \dim(V(\mathbf{w}))\partial_j(V(\mathbf{v})) - \sum k_l \partial_j(V(\mathbf{t}_l)).$$

Finally, in the last short `for` loop, $\partial_j(\lambda^2(\widetilde{\rho}_i))$, represented by `der`$j$, is computed by forming the appropriate combination of the derivatives of the terms of $\lambda^2(\widetilde{\rho}_i)$. These terms are in the polynomial `ext`, and the derivatives are incorporated as the coefficients in `pder`j.

This program does not perform the subtraction of $\dim(\rho_i)\widetilde{\rho}_i$, which is the last step in obtaining $L(\lambda^2\widetilde{\rho}_i)$. (See (2.4).) It was more convenient to do this in the `Maple` program which was the next step in our analysis. No doubt, this `LiE` program could be written more efficiently with arrays, but the author felt more comfortable with it in the given form.

8. ANALYSIS OF F_4 AND E_7 AT THE PRIME 3

When the analysis of Sections 2 and 3 is performed for F_4 at the prime 3, we find that its Adams operations are isomorphic to those of $B(11, 15) \times B_5(3, 23)$, where the second factor is an S^3-bundle over S^{23} with attaching map α_5. Since F_4 is known to be 3-equivalent to $B(11, 15) \times H$, where H is Harper's torsion H-space, this suggests that there is an isomorphism of Adams modules

$$PK^1(H)_{(3)} \approx PK^1(B_5(3, 23))_{(3)}$$

at the prime 3, or, more generally,

$$PK^1(H_p)_{(p)} \approx PK^1(B_{p+2}(3, 3 + q(p+2)))_{(p)},$$

for any odd prime p. This is implied by a result of Kono ([24]) on Chern character in $K^*(H_p)$. However, its implication for v_1-periodic homotopy groups does not quite agree with those determined for H_3 in [5] or H_p in [15].

The groups $v_1^{-1}\pi_*(B_{p+2}(3, 3 + q(p + 2)))$ computed in Proposition 5.5 agree with those computed by the UNSS in [10, 2.1]. They agree for the most part with the

groups $v_1^{-1}\pi_*(H_p)$ determined in [15, 3.2]. The calculation in Proposition 5.5 gives

$$v_1^{-1}\pi_{qi+2}(H_p) \approx v_1^{-1}\pi_{qi+1}(H_p) \approx \mathbf{Z}/p^{\min(p^2+p,1+\nu_p(i-p-2-\alpha p^{p^2+p-2}))},$$

with $\alpha = p + 2$, while that in [15, 3.2] is the same with $\alpha = 1$. This discrepancy led the author to find a small mistake in the delicate argument presented in [15] (for any odd prime p) and in [5] (for $p = 3$). The mistake in [5] occurs in the fifth bulleted item on page 300. The correct exponent of v_1 there is not 3, but rather a large number e which is not a multiple of 3. Then $(\eta_R(v_1^e) - v_1^e) \otimes h_2^3$ does not desuspend to S^5, as would have been the case if $e = 3$. This affects [5, (2.18)], causing the 1 at the end of it to be changed to a 2. The same mistake occurs in [15] in the second half of page 91; in this case the exponent of v_1 was overlooked entirely. This was one of the rare places where exponents of v_1 are consequential.

We have used the methods of this paper to check the result for $v_1^{-1}\pi_{\mathrm{ev}}(E_7; 3)$ obtained in [16]. We obtain complete agreement with the results presented there. In that paper, it was stated that if j is odd and $j \equiv 5$ or $8 \bmod 9$, then

$$v_1^{-1}\pi_{2j}(E_7) \approx v_1^{-1}\pi_{2j-1}(E_7) \approx \mathbf{Z}/p \oplus \mathbf{Z}/3^{\min(19,\nu(j-17-2\delta\cdot3^{13})+4)},$$

with $\delta = 2$, 5, or 8. The calculation based upon the methods of this paper shows that $\delta = 5$. Also, in [16] it was stated that if j is odd and $\nu_3(j - 11) \geq 10$, then

$$v_1^{-1}\pi_{2j}(E_7) \approx \mathbf{Z}/3^3 \oplus \mathbf{Z}/3^{12} \text{ or } \mathbf{Z}/3^4 \oplus \mathbf{Z}/3^{11}.$$

By the methods of this paper, we can show that the first of these splittings is the valid one.

References

[1] J. F. Adams, *Vector fields on spheres*, Annals of Math **75** (1962) 603-632.

[2] ―――――, *On the groups J(X)-II*, Topology **3** (1965) 137-171.

[3] ―――――, *On the groups J(X)-IV*, Topology **5** (1966) 21-71.

[4] C. M. Andersen, *Clebsch-Gordan series for symmetrized tensor products*, Jour Math Physics **8** (1967) 988-997.

[5] M. Bendersky and D. M. Davis, *3-primary v_1-periodic homotopy groups of F_4 and E_6*, Trans Amer Math Soc **344** (1994) 291-306.

[6] ―――――, *2-primary v_1-periodic homotopy groups of $SU(n)$*, Amer Jour Math **114** (1991) 465-494.

[7] ―――――, *The 1-line of the K-theory Bousfield-Kan spectral sequence for $Spin(2n + 1)$*, Contemporary Math **279** (2001) 37-56.

[8] ―――――, *A stable approach to an unstable homotopy spectral sequence*, to appear.

[9] M. Bendersky, D. M. Davis, and M. Mahowald, *v_1-periodic homotopy groups of $Sp(n)$*, Pacific Journal of Math **170** (1995) 319-378.

[10] M. Bendersky, D. M. Davis, and M. Mimura, *v_1-periodic homotopy groups of exceptional Lie groups: torsion-free cases*, Trans Amer Math Soc **333** (1992) 115-135.

[11] M. Bendersky and R. D. Thompson, *The Bousfield-Kan spectral sequence for periodic homology theories*, Amer Jour Math **122** (2000) 599-635.

[12] A. K. Bousfield, *The K-theory localizations and v_1-periodic homotopy groups of H-spaces*, Topology **38** (1999) 1239-1264.

[13] ———, *On Λ-rings and the K-theory of infinite loop spaces*, K-Theory **10** (1996) 1-30.

[14] D. M. Davis, *v_1-periodic homotopy groups of $SU(n)$ at odd primes*, Proc London Math Society **43** (1991) 529-544.

[15] ———, *Equivalences of some v_1-telescopes*, Contemporary Math **188** (1995) 81-92.

[16] ———, *3-primary v_1-periodic homotopy groups of E_7*, New York Jour Math **4** (1998) 185-221.

[17] D. M. Davis and M. Mahowald, *Some remarks on v_1-periodic homotopy groups*, Proc Adams Symposium, London Math Soc Lecture Notes Series **176** (1992) 55-72.

[18] ———, *Three contributions to the homotopy theory of the exceptional Lie groups G_2 and F_4*, Jour Math Soc Japan **43** (1991) 661-671.

[19] H. Hamanaka and S. Hara, *The mod 3 homology of the space of loops on the exceptional Lie groups and the adjoint action*, Jour Math Kyoto Univ **37** (1997) 441-453.

[20] J. Harper, *H-spaces with torsion*, Mem Amer Math Soc **223** (1979).

[21] L. Hodgkin, *On the K-theory of Lie groups*, Topology **6** (1967) 1-36.

[22] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag (1972).

[23] A. U. Klimyk, *Decomposition of a tensor product of irreducible representations of a semisimple Lie algebra into a direct sum of irreducible representations*, Amer Math Society Translations **76** (1968) 63-73.

[24] A. Kono, *On Harper's mod p H-space of rank 2*, Proc Royal Society Edinburgh **118** (1991) 75-78.

[25] A. Kono and M. Mimura, *Cohomology operations and the Hopf algebra structures of the compact exceptional Lie groups E_7 and E_8*, Proc London Math Soc **35** (1977) 345-358.

[26] M. Mimura, G. Nishida, and H. Toda, *Mod p decomposition of compact Lie groups*, Publ RIMS Kyoto Univ **13** (1977) 627-680.

[27] M. A. A. van Leeuwen, `LiE`, *a software package for Lie group computations*, `http://wallis.univ-poitiers.fr/∼maavl/LiE`.

[28] T. Watanabe, *Chern characters on compact Lie groups of low rank*, Osaka Jour Math **22** (1985) 463-488.

[29] ———, *The Chern character homomorphism of the compact simply connected exceptional group E_6*, Osaka Jour Math **28** (1991) 663-681.

[30] ———, *Adams operations in the connective K-theory of compact Lie groups*, Osaka Jour Math **23** (1986) 617-632.

[31] C. Wilkerson, *Self-maps of classifying spaces*, Springer Verlag Lecture Notes
     in Math **418** (1974) 51-57.

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