BP-HOMOLOGY OF ELEMENTARY ABELIAN 2-GROUPS: BP-MODULE STRUCTURE

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Abstract. We determine the BP_*-module structure, mod higher filtration, of the main part of the BP-homology of elementary abelian 2-groups. The action is related to symmetric polynomials and to Dickson invariants.

1. Introduction and results

Let $BP_*(-)$ denote Brown-Peterson homology localized at 2. Its coefficient groups BP_* are a polynomial algebra over $\mathbb{Z}_{(2)}$ on classes v_j, $j \geq 1$, of grading $2(2^j-1)$. Let $v_0 = 2$. As was done in [6] and [8], we consider $\bigotimes^k_{BP_*} BP_*(B\mathbb{Z}/2)$, which is a BP_*-direct summand of $BP_*(B(\mathbb{Z}/2)^k)$. We determine the BP_*-module structure of $\bigotimes^k_{BP_*} BP_*(B\mathbb{Z}/2)$ modulo terms which are more highly divisible by v_j's. Information about the action of v_0 was applied to problems in topology in [2] and [9]. In the forthcoming paper [3], we apply it to another problem, higher topological complexity of real projective spaces. In Theorem 1.7 of the current paper, we obtain complete explicit information about the v_0-action (mod higher filtration). In Theorem 1.1, we determine the action of all v_j's as quotients of symmetric polynomials, and in Theorem 1.3 and Corollary 1.6 we give explicit formulas as symmetric polynomials in certain families of cases. In Section 4, we discuss relationships of our symmetric polynomials with the Dickson invariants.

Now we explain this more explicitly. There are BP_*-generators $z_i \in BP_{2i-1}(B\mathbb{Z}/2)$ for $i \geq 1$, and $\bigotimes^k_{BP_*} BP_*(B\mathbb{Z}/2)$ is spanned as a BP_*-module by classes $z_I = z_{i_1} \otimes \cdots \otimes z_{i_k}$ for $I = (i_1, \ldots, i_k)$ with $i_j \geq 1$. Let Z_k denote the graded set consisting of all such classes z_I. It was proved in [6, Thm 3.2] that $\bigotimes^k_{BP_*} BP_*(B\mathbb{Z}/2)$ admits

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a decreasing filtration by BP_*-submodules F_s such that, for $s \geq 0$, the quotient F_s/F_{s+1} is a vector space over the prime field $\mathbb{F}_2$ with basis all classes $(v_k^{t_k} v_{k+1}^{t_{k+1}} \cdots)z_I$ with $z_I \in Z_k$, $t_i \geq 0$, and $\sum t_i = s$.

Define an action of $\mathbb{F}_2[x_1, \dots, x_k]$ on the $\mathbb{F}_2$-vector space with basis Z_k by

$$x_1^{e_1} \cdots x_k^{e_k} \cdot z_I = z_{I-E},$$

where $I-E = (i_1-e_1, \dots, i_k-e_k)$; here, by convention, $z_J = 0$ if any entry of J is ≤ 0. For positive integers $t_1, \dots, t_r$, let $m_{t_1,\dots,t_r}$ denote the monomial symmetric polynomial in $x_1, \dots, x_k$, the smallest symmetric polynomial containing the monomial $x_1^{t_1} \cdots x_r^{t_r}$. Over $\mathbb{F}_2$, if $r = k$ and the t_i are distinct, it equals the Vandermonde determinant

$$\begin{vmatrix} x_1^{t_1} & \cdots & x_1^{t_k} \\ & \vdots & \\ x_k^{t_1} & \cdots & x_k^{t_k} \end{vmatrix}.$$

Our first theorem determines the action of v_j, $0 \leq j \leq k-1$, from F_s/F_{s+1} to F_{s+1}/F_{s+2}, as a ratio of monomial symmetric polynomials in $x_1, \dots, x_k$. Note that k is fixed throughout, and we are always dealing with polynomials over $\mathbb{F}_2$. This theorem will be proved in Section 2.

Theorem 1.1. *If F_s is as above, and $0 \leq j \leq k-1$, the action of v_j from F_s/F_{s+1} to F_{s+1}/F_{s+2} is multiplication by $\sum\limits_{\ell \geq k} v_\ell p_{\ell,j}$, where*

$$p_{\ell,j} = \frac{m_{2^0,\dots,\widehat{2^j},\dots,2^{k-1},2^\ell}}{m_{2^0,\dots,2^{k-1}}}. \tag{1.2}$$

(The $\widehat{2^j}$ notation denotes omission.) Moreover, $p_{\ell,j}$ is a symmetric polynomial, mod 2.

It is not *a priori* clear that the quotient on the right hand side of (1.2) should be a polynomial mod 2. In fact, if the 2^ℓ there is a replaced by a non-2-power and $k \geq 3$, then the ratio is not a polynomial mod 2.

We have obtained explicit polynomial formulas for $p_{\ell,j}$ in several cases. These will be proved in Section 3. The first is the complete solution when $k = 3$.

Theorem 1.3. *If $k = 3$ and $\ell \geq 3$, then*

$$\begin{aligned} p_{\ell,0} &= \sum_{\substack{i \geq j \geq k > 0 \\ i+j+k=2^\ell-1}} \tbinom{j+k}{k} m_{i,j,k} \\ p_{\ell,1} &= \sum_{\substack{i \geq j > 0 \\ i+j=2^\ell-2}} (1+j) m_{i,j,0} + \sum_{\substack{i \geq j \geq k > 0 \\ i+j+k=2^\ell-2}} \left(1 + \tbinom{j+k}{k-1} + \tbinom{j+k+1}{k+1}\right) m_{i,j,k} \\ p_{\ell,2} &= \sum_{\substack{i \geq j \geq k \geq 0 \\ i+j+k=2^\ell-4}} \left(1 + \tbinom{j+k+2}{k+1}\right) m_{i,j,k}. \end{aligned}$$

Incorporating Theorem 1.3 into Theorem 1.1 gives the v_0-, v_1-, and v_2-action, mod higher filtration, in $BP_*(B\mathbb{Z}/2) \otimes_{BP_*} BP_*(B\mathbb{Z}/2) \otimes_{BP_*} BP_*(B\mathbb{Z}/2)$. For example, v_0 acts as

$$(1.4) \quad v_3 m_{4,2,1} + v_4(m_{12,2,1} + m_{10,4,1} + m_{8,6,1} + m_{9,4,2} + m_{8,5,2} + m_{8,4,3}) + \cdots,$$

where the omitted terms involve v_ℓ for $\ell \geq 5$.

We have also obtained the explicit polynomial formula for (1.2) for any k if $\ell = k$.

Theorem 1.5. *If $\ell = k$, then $p_{\ell,j} = p_{k,j}$ equals the sum of all monomials of degree $2^k - 2^j$ in $x_1, \ldots, x_k$ in which all nonzero exponents are 2-powers. Here $0 \leq j \leq k-1$.*

Theorem 1.5 gives the formula for the v_k-component of the BP_*-module structure, modulo higher filtration, of $\bigotimes_{BP_*}^k BP_*(B\mathbb{Z}/2)$. It is complete information, mod higher filtration, for $BP\langle k\rangle$ homology. Johnson-Wilson homology $BP\langle k\rangle$, introduced in [7], has coefficients $\mathbb{Z}_{(2)}[v_1, \ldots, v_k]$. By [6, Thm 3.2] and [8, Thm 1.1], as an abelian group $\bigotimes_{BP\langle k\rangle_*}^k BP\langle k\rangle_*(B\mathbb{Z}/2)$ has basis $\{v_k^j z_I : \ j \geq 0, \ z_I \in Z_k\}$.

Corollary 1.6. *In $\bigotimes_{BP\langle k\rangle_*}^k BP\langle k\rangle_*(B\mathbb{Z}/2)$, for $0 \leq j \leq k-1$,*

$$v_j \cdot z_I \equiv v_k \sum_E z_{I-E}$$

mod higher filtration, where $E = (e_1, \ldots, e_k)$ ranges over all k-tuples such that all nonzero e_j are 2-powers, and $\sum e_j = 2^k - 2^j$.

This generalizes [8, Cor 2.7], which says roughly that v_0 acts as $v_k m_{2^{k-1}, 2^{k-2}, \ldots, 1}$.

Finally, our most elaborate, and probably most useful, explicit calculation is given in the following result, which gives the complete formula for the v_0-action, mod higher filtration. This is useful since v_0 corresponds to multiplication by 2.

Theorem 1.7. *In $\bigotimes_{BP_*}^k BP_*(B\mathbb{Z}/2)$, v_0 acts as $\displaystyle\sum_{\ell\ge k} v_\ell\cdot p_{\ell,0}$ mod higher filtration, where*

$$p_{\ell,0}=\sum_f\prod_{i=0}^{\ell-1}x_{f(i)}^{2^i},$$

where f ranges over all surjective functions $\{0,\ldots,\ell-1\}\to\{1,\ldots,k\}$. Equivalently, $p_{\ell,0}=\sum m_{\|S_1\|,\ldots,\|S_k\|}$, where the sum ranges over all $\|S_1\|>\cdots>\|S_k\|$ with $S_1,\ldots,S_k$ a partition of $\{1,2,4,\ldots,2^{\ell-1}\}$ into k nonempty subsets. Here $\|S\|$ is the sum of the elements of S.

See (1.4) for an explicit example of $p_{3,0}$ and $p_{4,0}$ when $k=3$. For example, the term $m_{10,4,1}$ in $p_{4,0}$ corresponds to $S_1=\{8,2\}$, $S_2=\{4\}$, and $S_3=\{1\}$, and this corresponds to the sum of all surjective functions $f:\{0,1,2,3\}\to\{1,2,3\}$ for which $f(3)=f(1)$.

We thank a referee for many useful suggestions. See especially Section 4.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $Q=\bigotimes_{BP_*}^k BP_*(B\mathbb{Z}/2)$. Let z_i and z_I be as in the second paragraph of the paper. By [6], Q is spanned by classes $(v_0^{t_0}v_1^{t_1}\cdots)z_I$ with only relations $\sum_{j\ge0}a_jz_{i-j}$ in any factor, where $a_j\in BP_{2j}$ are coefficients in the [2]-series. By [11, 3.17], these satisfy, mod $(v_0,v_1,\ldots)^2$,

$$a_j\equiv\begin{cases}v_i & j=2^i-1,\ i\ge0\\ 0 & j+1 \text{ not a 2-power.}\end{cases}$$

Let F_s denote the ideal $(v_0,v_1,\ldots)^sQ$. Then F_s/F_{s+1} is spanned by all $(v_0^{t_0}v_1^{t_1}\cdots)z_I$ with $\sum t_j=s$, with relations

$$\sum_{j\ge0}v_jz_{i-(2^j-1)}=0 \tag{2.1}$$

in each factor. As proved in [6, Thm 3.2] (see also [8, 2.3]), this leads to an $\mathbb{F}_2$-basis for F_s/F_{s+1} consisting of all $(v_k^{t_k}v_{k+1}^{t_{k+1}}\cdots)z_I$ with $\sum t_j=s$.

We claim that if $z_I\in F_0$ and $0\le j\le k-1$, then we must have

$$v_jz_I=\sum_{\ell\ge k}v_\ell p_{\ell,j}z_I, \tag{2.2}$$

where $p_{\ell,j}$ is a symmetric polynomial in variables $x_1, \ldots, x_k$ of degree $2^\ell - 2^j$, acting on z_I by decreasing subscripts as described in the third paragraph of the paper. That the action is symmetric and uniform is due to the uniform nature of the relations (2.1). That it never increases subscripts of z_i is a consequence of naturality: there are inclusions $\bigotimes_{BP_*} BP_*(RP^{2n_i}) \to \bigotimes_{BP_*}^k BP_*(B\mathbb{Z}/2)$ in which the only z_I in the image are those with $i_t \le n_t$ for all t, and the v_j-actions are compatible.

Note that (2.1) can be interpreted as saying that, for any $i \in \{1, \ldots, k\}$,

$$\sum_{j \ge 0} v_j x_i^{2^j - 1} = 0. \tag{2.3}$$

Since the v_ℓ-components are independent if $\ell \ge k$, and (2.2) says that for $j < k \le \ell$ the v_ℓ-component of the v_j-action is given by the (unknown) polynomial $p_{\ell,j}$, we obtain the equation

$$\sum_{j=0}^{k-1} p_{\ell,j} x_i^{2^j - 1} = x_i^{2^\ell - 1}$$

for any $i \in \{1, \ldots, k\}$ and $\ell \ge k$. After multiplying the ith equation by x_i, we obtain the system

$$\begin{bmatrix} x_1 & x_1^2 & x_1^4 & \cdots & x_1^{2^{k-1}} \\ & & \vdots & & \\ x_k & x_k^2 & x_k^4 & \cdots & x_k^{2^{k-1}} \end{bmatrix} \begin{bmatrix} p_{\ell,0} \\ \vdots \\ p_{\ell,k-1} \end{bmatrix} = \begin{bmatrix} x_1^{2^\ell} \\ \vdots \\ x_k^{2^\ell} \end{bmatrix}, \tag{2.4}$$

whose solution as (1.2) is given by Cramer's Rule. Our argument shows that the components $p_{\ell,j}$ of the solution are polynomials, mod 2.

The ratios on the RHS of (1.2) can also be shown to be polynomials by the following algebraic argument, provided by the referee. Let V denote the $\mathbb{F}_2$-vector space with basis $x_1, \ldots, x_k$. The denominator $m_{1,2,\ldots,2^{k-1}}$ in (1.2) equals the product of the nonzero elements v of V. We show that a Vandermonde determinant D in $x_1, \ldots, x_k$ with distinct 2-power exponents 2^{t_j} is divisible by each v in the unique factorization domain $\mathbb{F}_2[x_1, \ldots, x_k]$, and hence is divisible by their product.

By induction on k and expansion along rows, the determinant is divisible by all elements except perhaps $\sum_{i=1}^{k} x_i$. Let $M_{k,j}$ denote the minor associated with $x_k^{2^{t_j}}$.

Replacing the last row by the sum of the others shows that

$$\sum_j M_{k,j}\sum_{i=1}^{k-1} x_i^{2^{t_j}}=0$$

since it is the determinant of a matrix with dependent rows. Thus

$$D=\sum_j M_{k,j}x_k^{2^{t_j}}=\sum_j M_{k,j}\sum_{i=1}^{k} x_i^{2^{t_j}}=\sum_j M_{k,j}\Bigl(\sum_{i=1}^{k} x_i\Bigr)^{2^{t_j}}$$

is divisible by $\sum_{i=1}^{k} x_i$.

The v_j-action formula on F_0 applies also on F_s by the nature of the module. ■

3. Proofs of explicit formulas for certain $p_{\ell,j}$

In this section, we prove Theorems 1.3, 1.5, and 1.7.

Proof of Theorem 1.3. Let $h_d(x_1,\dots,x_r)$ denote the complete homogeneous polynomial of degree d. With $k=3$, after a few row operations, (2.4) reduces to

$$\begin{bmatrix}1 & x_1 & x_1^3\\ 0 & 1 & h_2(x_1,x_2)\\ 0 & 0 & x_1+x_2+x_3\end{bmatrix}\begin{bmatrix}p_{\ell,0}\\ p_{\ell,1}\\ p_{\ell,2}\end{bmatrix}=\begin{bmatrix}x_1^{2^\ell-1}\\ h_{2^\ell-2}(x_1,x_2)\\ h_{2^\ell-3}(x_1,x_2,x_3)\end{bmatrix}.$$

Using Pascal's formula, one easily verifies, mod 2,

$$h_{n+1}(x_1,x_2,x_3)\equiv(x_1+x_2+x_3)\sum_{k,j}\left(\tbinom{n+2-k}{j+1}-1\right)x_1^{n-j-k}x_2^jx_3^k.$$

Since $\binom{2^\ell-2-k}{j+1}\equiv\binom{j+k+2}{j+1}$, the result for $p_{\ell,2}$ follows.

Now we have

$$\begin{aligned}p_{\ell,1}&=h_{2^\ell-2}(x_1,x_2)-h_2(x_1,x_2)p_{\ell,2}\\ &=\sum x_1^ix_2^{2^\ell-2-i}+(x_1^2+x_1x_2+x_2^2)\sum_{\substack{i\ge j\ge k\ge 0\\ i+j+k=2^\ell-4}}\left(1+\tbinom{j+k+2}{k+1}\right)m_{i,j,k}.\end{aligned}$$

If $k>0$, the coefficient of $m_{i,j,k}$ in this is

$$\left(1+\tbinom{j+k+2}{k+1}\right)+\left(1+\tbinom{j+k+1}{k+1}\right)+\left(1+\tbinom{j+k}{k+1}\right),$$

which equals the claimed value. If $k=0$ and $j>0$, there is an extra 1 from the $\sum x_1^i x_2^{2^\ell-2-i}$, and we obtain $\binom{j+2}{1}+\binom{j+1}{1}+\binom{j}{1}\equiv 1+j$, as desired. The coefficient of $m_{2^\ell-4,0,0}$ is easily seen to be 0.

Finally, we obtain $p_{\ell,0}$ from $x_1^{2^\ell-1}+x_1p_{\ell,1}+x_1^3p_{\ell,2}$. The coefficient of $m_{i,j,0}$ in this is $(1+j)+(1+\binom{j+2}{1})=0$, as desired. If $k>0$, the coefficient of $m_{i,j,k}$ is $(1+\binom{j+k}{k-1}+\binom{j+k+1}{k+1})+(1+\binom{j+k+2}{k+1})\equiv\binom{j+k}{k}$, as desired. ■

Proof of Theorem 1.5. It suffices to show that

$$\sum_{i=1}^{k} x_1^{2^{i-1}} g_{2^k-2^{i-1}} = x_1^{2^k}, \tag{3.1}$$

where g_m is the sum of all monomials in $x_1,\ldots,x_k$ of degree m with all nonzero exponents 2-powers. (Other rows are handled equivalently.)

The term $x_1^{2^k}$ is obtained once, when $i=k$. The only monomials obtained in the LHS of (3.1) have their x_i-exponent a 2-power for $i>1$, while their x_1-exponent may be a 2-power or the sum of two distinct 2-powers. A term of the first type, $x_1^{2^i}x_2^{2^{t_2}}\cdots x_k^{2^{t_k}}$ with $\sum 2^{t_i}>0$, can be obtained from either the ith term in (3.1) or the $(i+1)$st. So its coefficient is 0 mod 2. A term of the second type, $x_1^{2^a+2^b}x_2^{t_2}\cdots x_k^{t_k}$, can also be obtained in two ways, either from $i=a+1$ or $i=b+1$. ■

Theorem 1.7 is an immediate consequence of the following proposition, which shows that, in $\mathbb{F}_2[x_1,\ldots,x_k]$,

$$m_{2^1,\ldots,2^{k-1},2^\ell}=m_{2^0,\ldots,2^{k-1}}\cdot\sum m_{\|S_1\|,\ldots,\|S_k\|},$$

with S_i as in Theorem 1.7 or Proposition 3.2.

Proposition 3.2. *For $\ell\ge k$, the only k-tuples $(n_1,\ldots,n_k)$ that can be decomposed in an odd number of ways as $n_i=s_i+t_i$ with $(t_1,\ldots,t_k)$ a permutation of $(1,2,4,\ldots,2^{k-1})$ and $s_i=\|S_i\|$, where $S_1,\ldots,S_k$ is a partition of $\{1,2,4,\ldots,2^{\ell-1}\}$ into k nonempty subsets, are the permutations of $(2,4,8,\ldots,2^{k-1},2^\ell)$.*

Proof. We will show that all

$$\begin{pmatrix} S_1 & \cdots & S_k\\ t_1 & \cdots & t_k\end{pmatrix} \tag{3.3}$$

as in the proposition can be grouped into pairs with equal column sums ($\|S_1\|+t_1,\dots,\|S_k\|+t_k$) except for permutations (by column) of

$$\begin{pmatrix} 2^0 & 2^1 & \cdots & 2^{k-2} & \{2^{k-1},\dots,2^{\ell-1}\} \\ 2^0 & 2^1 & \cdots & 2^{k-2} & 2^{k-1} \end{pmatrix}. \tag{3.4}$$

It is easy to see that (3.4) is the only matrix (3.3) with its column sum.

Let M be a matrix (3.3), and let $K=\{2^0,\dots,2^{k-1}\}$. Define $f:K\to K$ by $f(x)=t_i$ if $x\in S_i$. Since $f^{i+1}K\subseteq f^iK$, there is a smallest nonnegative integer N such that $f^{N+1}K=f^NK$; i.e., with $T:=f^NK$, $f|T$ is an automorphism of T.

Case 1: $f|T\ne 1_T$. We pair M with the matrix obtained by interchanging x and $f(x)$ in all columns with $t_i\in T$. Note that this preserves column sums and is involutive, in the sense that the new matrix is also of Case 1 type, and would lead to M. For example,

$$\begin{pmatrix} 2^0 & \{2^1,2^3\} & 2^2 \\ 2^1 & 2^0 & 2^2 \end{pmatrix} \text{ is paired with } \begin{pmatrix} 2^1 & \{2^0,2^3\} & 2^2 \\ 2^0 & 2^1 & 2^2 \end{pmatrix}.$$

Case 2: $f|T=1_T$. Let $2^i\in T$ be minimal such that the S_j above it in (3.3) *strictly* contains $\{2^i\}$. Such an i must exist since either $T=K$ or else some $\ell\in K-T$ must satisfy $f(\ell)\in T$.

Case 2a: $i<k-1$. Then

$$\begin{pmatrix} \cdots & \{2^i,D\} & \cdots & E & \cdots \\ \cdots & 2^i & \cdots & 2^{i+1} & \cdots \end{pmatrix} \text{ is paired with } \begin{pmatrix} \cdots & D & \cdots & \{2^i,E\} & \cdots \\ \cdots & 2^{i+1} & \cdots & 2^i & \cdots \end{pmatrix}.$$

Here D and E represent nonempty collections of 2-powers.

Case 2b: $i=k-1$. Let S_v be the set above 2^{k-1} in (3.3). If S_v contains $\{2^i:\ k-1\le i\le \ell-1\}$, then the matrix must be of the form (3.4), since f must be bijective, and hence $T=K$ and $f=1_K$. Otherwise, let 2^e be the smallest 2-power $\ge 2^k$ not in S_v. There is a sequence $2^{i_1},\dots,2^{i_r}$ such that 2^{i_1} lies below 2^e in M, $f(2^{i_j})=2^{i_{j+1}}$ for $1\le j<r$, and $2^{i_r}\in S_v$. This sequence of 2^{i_j}'s must eventually be in S_v because otherwise it would have a cycle, and be in Case 1. The matrix M is paired with one in which all the 2^{i_j}'s are moved up or down within their column, while the 2^j's with $k-1\le j\le e$ are interchanged between the columns containing the 2^e and the 2^{k-1}, with other entries remaining fixed. We illustrate with a case $r=2$, $e=k+2$.

$$\begin{pmatrix} 2^{k+2} & \cdots & 2^{t_1} & \cdots & \{2^{t_2},2^{k-1},2^k,2^{k+1}\} \\ 2^{t_1} & \cdots & 2^{t_2} & \cdots & 2^{k-1} \end{pmatrix} \leftrightarrow \begin{pmatrix} \{2^{t_1},2^{k-1},2^k,2^{k+1}\} & \cdots & 2^{t_2} & \cdots & 2^{k+2} \\ 2^{k-1} & \cdots & 2^{t_1} & \cdots & 2^{t_2} \end{pmatrix}.$$

■

4. Relations with Dickson invariants

In this section, we discuss the relationship between our polynomials $p_{\ell,j}$ and the Dickson invariants. Most of the results in this section were suggested by a referee.

Let V be an $\mathbb{F}_2$-vector space with basis $x_1, \ldots, x_k$, and $S(V)$ its symmetric algebra. The general linear group $\mathrm{GL}(V)$ acts on $S(V)$, and the ring of invariant elements is called the 2-primary Dickson algebra D_k. Dickson showed in [4] that D_k is a polynomial algebra on classes c_j of grading $2^k - 2^j$ for $0 \le j \le k-1$. We suppress the usual k from the subscript, as we did with our p's, since it is fixed throughout this paper.

If M is a Vandermonde determinant in $x_1, \ldots, x_k$ with distinct 2-power exponents, then M is invariant under the action of $\mathrm{GL}(V)$. This is easily proved using linearity of determinants and that $(\sum \alpha_i x_i)^{2^t} = \sum \alpha_i x_i^{2^t}$. Since our polynomials $p_{\ell,j}$ in (1.2) are ratios of Vandermonde determinants with distinct 2-power exponents, they are elements of D_k, and one might seek to express them in terms of the generators c_j.

Our first result is that our polynomials $p_{k,j}$ (i.e., those with $\ell = k$) are exactly the generators c_j.

Proposition 4.1. *For $0 \le j \le k-1$, $p_{k,j} = c_j$.*

Proof. By [10, Prop 1.3a], $c_j = \dfrac{m_{2^0,\ldots,\widehat{2^j},\ldots,2^{k-1},2^k}}{m_{2^0,\ldots,2^{k-1}}}$, which by (1.2) equals $p_{k,j}$. ■

The following corollary is now immediate from Theorem 1.5.

Corollary 4.2. *The Dickson invariant usually called $c_{k,j}$ over $\mathbb{F}_2$ is the sum of all monomials of degree $2^k - 2^j$ in $x_1, \ldots, x_k$ in which all nonzero exponents are 2-powers.*

This result was certainly known to some, but we could not find it explicitly stated in the literature. One place that essentially says it is [1, Prop 3.6(c)].

Some of our elements $p_{\ell,j}$ are related to one another in the following way.

Proposition 4.3. *For $\ell \ge k+1$, we have $p_{\ell,0} = c_0 p_{\ell-1,k-1}^2$. In particular, $p_{k+1,0} = c_0 c_{k-1}^2$.*

Proof. The denominator in (1.2) equals c_0, so we have

$$c_0 p_{\ell,0} = m_{2^1,\dots,2^{k-1},2^\ell} = m^2_{2^0,\dots,2^{k-2},2^{\ell-1}} = c_0^2 p^2_{\ell-1,k-1}.$$

The second part follows from Proposition 4.1. ■

There is an action of the mod-2 Steenrod algebra on $S(V)$ and on D_k, and the following complete formula was obtained in [5].

Proposition 4.4. ([5]) *In the Dickson algebra D_k, for $0 \le s \le k-1$,*

$$\mathrm{Sq}^i\, c_s = \begin{cases} c_r & i = 2^s - 2^r \\ c_r c_t & i = 2^k - 2^t + 2^s - 2^r,\ r \le s < t \\ c_s^2 & i = 2^k - 2^s \\ 0 & \textit{otherwise.} \end{cases}$$

Without using that formula, we can easily obtain the following result.

Proposition 4.5. *For $0 \le j \le k-1$,*

$$\mathrm{Sq}^{2^i}\, p_{\ell,j} = \begin{cases} p_{\ell,j-1} & i = j-1 \\ 0 & i \ne j-1,\ i < k-1. \end{cases}$$

Proof. For $i < k-1$, $\mathrm{Sq}^{2^i}(m_{2^0,\dots,2^{k-1}}) = 0$ since each term with factor $\mathrm{Sq}^{2^i}(x_s^{2^i})x_t^{2^{i+1}}$ is paired with an equal term $x_s^{2^{i+1}}\,\mathrm{Sq}^{2^i}(x_t^{2^i})$. Using the Adem relations, it follows that $\mathrm{Sq}^n(m_{2^0,\dots,2^{k-1}}) = 0$ for $0 < n < 2^{k-1}$. Similarly, for $0 < i < k-1$,

$$\mathrm{Sq}^{2^i}(m_{2^0,\dots,\widehat{2^j},\dots,2^{k-1},2^\ell}) = \begin{cases} m_{2^0,\dots,\widehat{2^{j-1}},\dots,2^{k-1},2^\ell} & i = j-1 \\ 0 & \text{otherwise.} \end{cases}$$

The result follows from applying the Cartan formula to

$$m_{2^0,\dots,2^{k-1}} p_{\ell,j} = m_{2^0,\dots,\widehat{2^j},\dots,2^{k-1},2^\ell}.$$

■

This result meshes nicely with the following one.

Proposition 4.6. *For $\ell \ge k$,*

$$p_{\ell+1,k-1} = \sum_j c_j\, \mathrm{Sq}^{2^\ell - 2^k + 2^j}\, p_{\ell,k-1}.$$

Proof. We have

(4.7)
$$\mathrm{Sq}^{2^\ell}(c_0 p_{\ell,k-1}) = \mathrm{Sq}^{2^\ell}(m_{2^0,\dots,2^{k-2},2^\ell}) = m_{2^0,\dots,2^{k-2},2^{\ell+1}} = c_0 p_{\ell+1,k-1}.$$

As a special case of Proposition 4.4, we have, for $i > 0$,

$$\mathrm{Sq}^i\, c_0 = \begin{cases} c_j c_0 & i = 2^k - 2^j \\ 0 & \text{otherwise.} \end{cases}$$

Applying the Cartan formula to the LHS of (4.7) and cancelling c_0 yields the result. ∎

In principle, iterating Propositions 4.5 and 4.6 enables us to obtain complete formulas expressing our polynomials $p_{\ell,j}$ in terms of the c_i's. For $\ell = k$, this was initiated in our Proposition 4.1. Here we do it for $\ell = k+1$ and $k+2$. For $\ell \geq k+3$, the formulas become unwieldy.

Theorem 4.8. *For* $0 \leq j \leq k-1$,

a. $p_{k+1,j} = c_{j-1}^2 + c_j c_{k-1}^2$;
b. $p_{k+2,j} = c_j c_{k-2}^4 + c_j c_{k-1}^6 + c_{j-1}^2 c_{k-1}^4 + c_{j-2}^4$.

Proof. (a). By Propositions 4.1, 4.6, and 4.4, we have

$$\begin{aligned} p_{k+1,k-1} &= \sum_j c_j\, \mathrm{Sq}^{2^j}\, p_{k,k-1} = \sum_j c_j\, \mathrm{Sq}^{2^j}\, c_{k-1} \\ &= c_{k-2}^2 + c_{k-1} \cdot c_{k-1}^2. \end{aligned}$$

Assume the result true for j. By Propositions 4.5 and 4.4,

$$\begin{aligned} p_{k+1,j-1} &= \mathrm{Sq}^{2^{j-1}}(c_{j-1}^2 + c_j c_{k-1}^2) \\ &= (\mathrm{Sq}^{2^{j-2}}\, c_{j-1})^2 + \sum_m (\mathrm{Sq}^{2^{j-1}-2m}\, c_j)(\mathrm{Sq}^m\, c_{k-1})^2 \\ &= c_{j-2}^2 + (\mathrm{Sq}^{2^{j-1}}\, c_j) c_{k-1}^2 \\ &= c_{j-2}^2 + c_{j-1} c_{k-1}^2, \end{aligned}$$

extending the induction.

(b). Applying Proposition 4.6 to part (a), we obtain

$$\begin{aligned} p_{k+2,k-1} &= \sum_j c_j \operatorname{Sq}^{2^k+2^j}(c_{k-2}^2+c_{k-1}^3) \\ &= \sum_j c_j(\operatorname{Sq}^{2^{k-1}+2^{j-1}} c_{k-2})^2 + \sum_{j,m} c_j(\operatorname{Sq}^{2^k+2^j-2m} c_{k-1})(\operatorname{Sq}^m c_{k-1})^2. \end{aligned}$$

Using Proposition 4.4, the first sum equals $c_{k-2}c_{k-3}^2c_{k-1}^2 + c_{k-1}c_{k-2}^4$, while the second equals

$$c_{k-3}^4 + c_{k-2}c_{k-1}^2c_{k-3}^2 + c_{k-2}^2c_{k-1}^4 + c_{k-1}^7.$$

Combining these yields the result for $j = k-1$. The result for arbitrary j follows by decreasing induction on j, similarly to part (a). ■

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