1. $1/(\frac{1}{3} - \frac{1}{4}) =$

2. Dick is 6 years older than Jane. Six years ago he was twice as old as she was. How old is Jane now?

3. A bicyclist riding against the wind averages 10 miles per hour in traveling from A to B, but with the wind averages 15 miles per hour in returning from B to A. How many miles per hour is his average speed for the trip?

4. What is the largest possible value for the sum of two fractions such that each of the four 1-digit prime numbers occurs as one of the numerators or denominators?

5. How many integers x in $\{1, 2, 3, \ldots, 99, 100\}$ satisfy that $x^2 + x^3$ is the square of an integer?

6. What is the number of real numbers x such that $25|x| = x^2 + 144$?

7. How many pairs (x, y) of positive integers satisfy $2x + 7y = 1000$?

8. A ladder is leaning against a house with its bottom 15 feet from the house. When its bottom is pulled 9 feet farther away from the house, the upper end slides 13 feet down. How many feet long is the ladder?

9. What is the sum of the three smallest prime numbers each of which is two more than a positive perfect cube?

10. Amy, Bob, and Chris each took a 6-question true-false exam. Their answers to the six questions in order were Amy:FFTTTT, Bob:TFFTTT, and Chris:TTFFTT. Amy and Bob each got 5 right. What is the most that Chris could have gotten right?

11. The two shortest sides of a right triangle have lengths 2 and $\sqrt{5}$. Let α be the smallest angle of the triangle. What is $\cos \alpha$?

12. From a point P on the circumference of a circle, perpendiculars PA and PB are dropped to points A and B on two mutually perpendicular diameters. If $AB = 8$, then what is the diameter of the circle?

13. How many 9's are in the decimal expansion of 99999989999^2? (This is the square of an 11-digit number.)

14. Let A be the point $(7, 4)$ and D be $(-5, -3)$. What is the length of the shortest path $ABCD$, where B is a point $(x, 2)$ and C is a point $(x, 0)$? This path consists of three connected segments, with the middle one vertical.

15. Simplify $\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$.

16. A square has its base on the x-axis, and one vertex on each branch of the curve $y = 1/x^2$. What is its area?

17. Which integer is closest to $\frac{1}{2}(\sqrt[3]{829} + \log_{10} 829)$?

18. In a 9-12-15 right triangle, a segment is drawn parallel to the hypotenuse one third of the way from the hypotenuse to the opposite vertex. Another segment is drawn parallel to the first segment one third of the way from it to the opposite vertex. Each segment is bounded by sides of the triangle on both ends. What is the area of the trapezoid inside the triangle between these two segments?

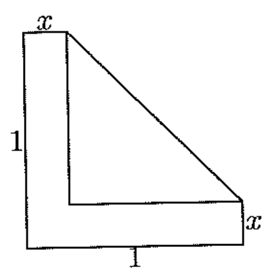
19. A rhombus has sides of length 10, and its diagonals differ by 4. What is its area?

20. What is the smallest positive integer k for which there exist integers $a > 1$ and $b > 1$ for which the correct simplification of $\sqrt{k}$ is $a\sqrt{b}$, and the correct simplification of $\sqrt[3]{k}$ is $b\sqrt[3]{a}$?

21. The diagonal BE in pentagon $ABCDE$ is the base of an isosceles triangle ABE and is also the base of an isosceles trapezoid $BCDE$. If $\angle C = 3\angle CBE$, express the number of degrees in angle A in terms of angle B ($= \angle ABC$). (Your answer should be an expression involving the letter B. Don't write "$A =$" on your answer sheet.)

22. In a group of five friends, the sums of the ages of each group of four of them are 124, 128, 130, 136, and 142. What is the age of the youngest?

23. In the figure below, if the area of the letter L part equals the area of the triangle, and the length of the base and of the height is 1, what is the length x of the ends of the L?



24. Let a, b, and c denote the three roots of $x^3 - 17x - 19$. What is the value of $a^3 + b^3 + c^3$?

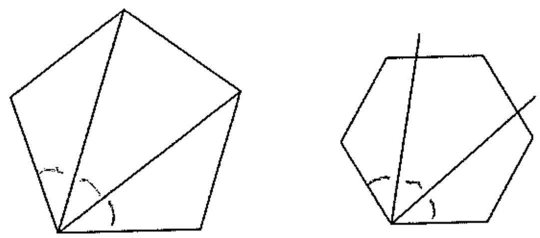
25. Determine all values of k such that the solution set of $|x - k| < 2$ is a subset of the solution set of $\frac{2x-1}{x+2} < 1$. (You may use interval notation or inequalities to express your answer.)

26. A rhombus of side length s has the property that there is a point on its longer diagonal such that the distances from that point to the vertices are 1, 1, 1, and s. What is the value of s?

27. Define $a * b = ab + b$. Find the set of all real numbers a such that the equation $x * (a * x) = -1/4$ has two distinct real solutions in x. You may use interval notation or inequalities.

28. Let Q denote the point $(4, 2)$. There are two points P on the circle $x^2 + y^2 = 2$ such that PQ is tangent to the circle. One of them, P_1, has integer components and the other, P_2, has fractional (non-integer) components. What are the components of P_2?

29. The angle trisectors of a regular pentagon intersect other vertices of the pentagon; i.e. they are diagonals of the pentagon. (See the left figure in the diagram below.) What is the smallest $n > 5$ such that the angle trisectors of a regular n-gon intersect other vertices of the n-gon? (The right side of the diagram below illustrates that $n = 6$ does not work.)



30. Let $\langle a_n \rangle$ denote an arithmetic sequence beginning $a_1 = 6$, $a_2 = 8$, $a_3 = 10$. Define a second sequence $\langle b_n \rangle$ by $b_1 = 3$ and $b_n = b_{n-1} + a_n$. Write an explicit expression for b_n in terms of n. (Just write the expression on your answer sheet. Don't write "$b_n =$".)

31. Let $d(n)$ denote the sum of the digits of n. It is well-known that $n - d(n)$ must be a multiple of 9. What is the smallest positive integer k such that k is a multiple of 9 and neither k nor $k + 9$ can be achieved as $n - d(n)$ for some n?

32. Let S denote the set of all (positive) divisors of 60^5. The product of all the numbers in S equals 60^e for some integer e. What is the value of e?

33. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$. A *special subset* of S is a subset B which satisfies the following three properties:
    a. B has exactly 8 elements;
    b. If $x \in S$ is even, then x is in B if and only if $x/2$ is in B;
    c. If $y \in S$ is odd, then y is in B if and only if $(y + 15)/2$ is in B.
    Which elements of S cannot be part of a special subset?

34. A semicircle is inscribed in a quadrilateral $ABCD$ in such a way that the midpoint of BC coincides with the center of the semicircle, and the diameter of the semicircle lies along a portion of BC. If $AB = 4$ and $CD = 5$, what is BC?

35. What are the two smallest positive integers n for which $360n$ has exactly 72 positive divisors?

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36. How many distinct triangles with positive integer sides have perimeter equal to 100? (Triangles are distinct if they are not congruent.)

37. Let $\{x\} = x - [x]$ denote the fractional part of x. For example, $\{3.4\} = .4$ and $\{4\} = 0$. Let $f(x) = \{3x/2\}$. How many solutions does the equation $f(f(f(f(x)))) = 0$ have in the interval $0 \le x < 1$?

38. Each side of a regular dodecagon $A_1A_2A_3 \ldots A_{12}$ has length 2. What is the area of the pentagon $A_1A_2A_3A_4A_5$?

39. Flip a fair coin repeatedly until the sequence HTH occurs. What is the probability that the sequence THTH has not yet occurred? (For example, the event whose probability we seek includes the sequence HHTH, but not the sequence TTHTH.)

40. In the rhombus in problem 26, what is the tangent of the acute angle of the rhombus? (Your answer should involve square roots, not trig functions.)

Solutions to 2006 Contest

Numbers in brackets are the number of people who answered the problem correctly, first out of the 36 people who scored at least 15, and then out of the other 151 people.

1. 12. [36,140.5] $1/(\frac{4}{12} - \frac{3}{12})$
2. 12. [35,137] $D = J + 6$. $D - 6 = 2(J - 6)$. Therefore, $J = 2J - 12$, so $J = 12$.
3. 12. [34,61] If d is the distance between A and B, then his time is $d/10 + d/15 = 5d/30 = d/6$. Thus his overall speed is $2d/(d/6) = 12$.
4. 31/6. [32,89] It must be $7/2 + 5/3$.
5. 9. [25,18] It is equivalent to say that $1 + x$ is a square. Then x can be $2^2 - 1, 3^2 - 1, \ldots, 10^2 - 1$.
6. 4. [34,42] Either $x^2 - 25x + 144 = 0$ or $x^2 + 25x + 144 = 0$. The first has solutions 9 and 16, while the second has solutions -9 and -16.
7. 71. [22,38] We must have $y = 2z$ and then $x + 7z = 500$. Now z can equal any number from 1 to 71, and x will be uniquely determined.
8. 25. [34,65.5] If ℓ denotes the length of the ladder and h the height to which it extends initially, then $\ell^2 = h^2 + 225$ and $\ell^2 = (h-13)^2 + 24^2$. Thus $26h = 169 + 576 - 225 = 520$. Hence $h = 20$. Then $\ell = 25$.
9. 159. [33,65] It is $3 + 29 + 127$, since 1, 27, and 125 are the first odd cubes.
10. 3. [28,114] The correct answers must have been either FFFTTT or TFTTTT. Either way, Chris got 3 right.
11. $\sqrt{5}/3$. [33,94] The hypotenuse is 3. The smaller angle has the larger cosine.
12. 16. [26,21] Let the circle be $x^2 + y^2 = r^2$ and the diameters be the x- and y-axes. If the coordinates of P are (x, y), then A and B are at $(x, 0)$ and $(0, y)$. Thus $AB = \sqrt{x^2 + y^2} = r$. Since $AB = 8$, the diameter is $2r = 16$.
13. 9. [11,13] The given number is $10^{11} - 10001$. It square is $10^{11}(10^{11} - 20002) + 100020001$. The number of 9's in this equals the number in $10^{11} - 20002 = 99999979998$.
14. 15. [7,5] Compress the horizontal band between the lines $y = 0$ and $y = 2$. Then the shortest distance between A and D is a straight line, and their displacement is 12 horizontal units and 5 vertical units (with the 2 compressed units omitted). This line has length 13, and the 2 must be added back.
15. 2. [21,11.5] $x^2 = 3 + 2\sqrt{2} + 3 - 2\sqrt{2} - 2\sqrt{9 - 8} = 6 - 2$.
16. $2\sqrt[3]{2}$ or $2^{4/3}$. [20,9] If one vertex is at $(x, 0)$, then the side of the square is $2x$ satisfying $2x = 1/x^2$. Thus $x^3 = 1/2$, and the area is $4x^2 = 2^{4/3}$.
17. 6. [30,51] Since $9^3 = 729$ and $10^3 = 1000$, $\sqrt[3]{829}$ is slightly greater than 9 (approx 9.3). Since $\log_{10} 100 = 2$ and $\log_{10} 1000 = 3$, $\log_{10} 829$ is slightly less than 3 (approx

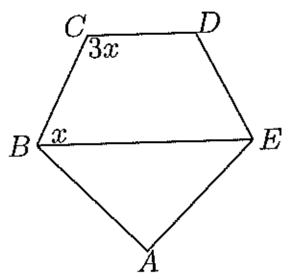
2.9). Thus the answer is approximately $(9+3)/2$. (Without making estimates, clearly the sum is between 11 and 13, and so 1/2 times the sum is definitely closest to 6.)

18. 40/3. [14,10] The area of the entire triangle is 54. The area of the triangle between the first line and opposite vertex is $\frac{4}{9} \cdot 54 = 24$. Four ninths of that area is removed when the second line is drawn. What remains is $\frac{5}{9} \cdot 24$.

19. 96. [28,31] The diagonals of a rhombus are perpendicular. If b denotes half the length of the shorter diagonal, then the rhombus is composed of four right triangles with bases b and $b+2$ and hypotenuse 10. Thus $b^2 + (b+2)^2 = 100$, hence $b^2 + 2b - 48 = 0$, so $b = 6$. The area of each of the triangles is $6 \cdot 8/2$.

20. 32. [35,40] We have $k = a^2 b$ and $k = ab^3$. Thus $a = b^2$. The smallest allowable values are $b = 2$, $a = 4$.

21. $270 - 2B$. [20,12] In the diagram below, $8x = 360$, so $x = 45$. Now $A + 2(B - 45) = 180$.



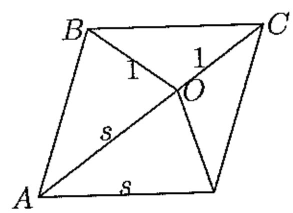
22. 23. [27,27] The sum of these five sums will include each of the people four times. Thus the sum of all their ages is 660/4=165. So the youngest is $165 - 142 = 23$.

23. $1 - \frac{1}{3}\sqrt{6}$. [20,17.5] Adjoining to the figure the reflection of the triangle across its hypotenuse yields a square. The area of the triangle must equal 1/3 times the area of the square, i.e. 1/3. Thus $\frac{1}{2}(1-x)^2 = \frac{1}{3}$ and so $1 - x = \sqrt{2/3}$.

24. 57. [5,2] Since $a^3 = 17a + 19$, etc., we have $a^3 + b^3 + c^3 = 17(a+b+c) + 57$. Since $x^3 - 17x - 19 = (x-a)(x-b)(x-c)$, $a + b + c$ equals the negative of the coefficient of x^2, which is 0.

25. $0 \le k \le 1$. [6,3] $\frac{2x-1}{x+2} - 1 < 0$ is equivalent to $\frac{x-3}{x+2} < 0$, which is equivalent to $-2 < x < 3$. We require that the open interval from $k - 2$ to $k + 2$ be contained in this.

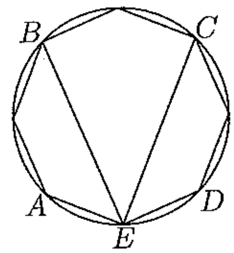
26. $(1+\sqrt{5})/2$. [7,0] In the diagram below, triangles BOC and ABC are similar. Thus $s : 1 = BC : CO = AC : BC = (s+1) : s$. Hence $s^2 = s + 1$. (Remark: this figure is a foundational figure in Penrose's aperiodic tilings.)



27. $a > 0$ or $a < -1$. [14,1] The equation becomes $(x+1)(a+1)x = -\frac{1}{4}$, or $(a+1)x^2 + (a+1)x + \frac{1}{4} = 0$. This has two distinct real solutions if $(a+1)^2 - (a+1) > 0$ hence $(a+1)a > 0$. Both factors are positive if $a > 0$ and negative if $a < -1$.

28. $(-1/5, 7/5)$. [5,0] Let (x, y) denote the components of either point P. Then $x^2 + y^2 = 2$, and $(0,0)$ and $(4,2)$ form the hypotenuse of a right triangle with other vertex at (x, y). Thus $x^2 + y^2 + (x-4)^2 + (y-2)^2 = 4^2 + 2^2$, and hence $2 = x^2 + y^2 = 4x + 2y$. Therefore $x^2 + (1-2x)^2 = 2$; i.e. $5x^2 - 4x - 1 = 0$, and so $x = 1$ or $-1/5$.

29. 8. [16,22] In general, n must be of the form $2 + 3k$. To see this, inscribe the polygon in a circle as below. If AEB, BEC, and CED trisect the angle, then the arcs AB, BC, and CD are equal. The angle must be chosen so that these three arcs can be filled in with k sides of the regular polygon.



30. $n^2 + 5n - 3$. [13.5,5] We have $a_n = 4 + 2n$. Since the differences in the b-sequence are linear in n, the sequence itself will be quadratic. If $b_n = An^2 + Bn + C$, then

$$An^2 + Bn + C = A(n-1)^2 + B(n-1) + C + 4 + 2n,$$

and hence $2An = A - B + 4 + 2n$. Therefore $A = 1$ and $B = 5$. Finally $1 + 5 + C = 3$.

31. 981. [8,1] If n is a 2-digit number a_1a_0, then $n - d(n) = 9a_1$. Thus the multiples of 9 up to 81 can be achieved as $n - d(n)$. If n is a 3-digit number $a_2a_1a_0$, then $(n - d(n))/9 = 11a_2 + a_1$. Since a_1 can go from 0 to 9, $(n - d(n))/9$ cannot equal $11 - 1, 22 - 1, \ldots, 99 - 1$, but can equal all other numbers in this range. The largest $(n - d(n))/9$ can be for a 3-digit number is $99 + 9$ and the smallest it can be for a 4-digit number is 111. Thus $9 \cdot 109$ is our desired answer.

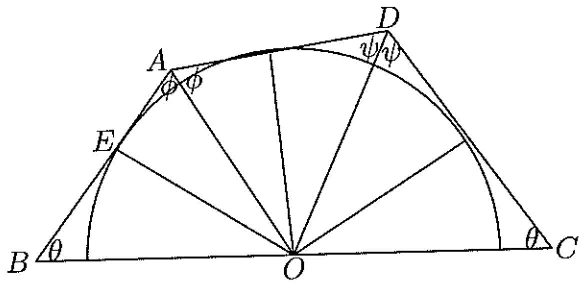
32. 990. [9,0] $60 = 2^2 \cdot 3 \cdot 5$. Hence $60^5 = 2^{10}3^55^5$. Thus

$$S = \{2^a3^b5^c : \ 0 \le a \le 10, \ 0 \le b, c \le 5\}.$$

Note S contains $11 \cdot 6 \cdot 6$ numbers. For each a, 2^a will occur as the 2-power in 36 of those numbers. Thus the exponent of 2 in the product is $36(0 + 1 + \cdots 10) = 36 \cdot 55$. Similarly the exponents of 3 and 5 in the product are each $66(0 + 1 + \cdots 5) = 66 \cdot 15$. This latter number is the desired exponent e. Note that $36 \cdot 55 = 2 \cdot 990$.

33. 5, 10. [9,19] The subsets of S which satisfy rules (b) and (c) are $\{1,2,4,8\}$, $\{3,6,9,12\}$, $\{7,11,13,14\}$, and $\{5,10\}$ and any union of these. To satisfy rule (a), we must take the union of two of the 4-element subsets just described.

34. $4\sqrt{5}$. [0,0] The indicated angles in the figure below are equal in pairs. (One way to see this for θ is to reflect triangle EBO through the origin.) Thus $360 = 2\theta + 2\phi + 2\psi$, and so $\angle BOA = \psi$ and $\angle COD = \phi$. Thus triangles BOA and CDO are similar, and hence $BO/AB = CD/OC$. If $x = BC$, this implies $(x/2)^2 = 4 \cdot 5$.



35. 28 and 35. [2,0] We must have $360n = 2^a 3^b 5^c 7^d$ with $a \geq 3$, $b \geq 2$, $c \geq 1$, and $d \geq 0$, and

$$(a+1)(b+1)(c+1)(d+1) = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3.$$

The possibilities for $(a+1, b+1, c+1, d+1)$ are (8,3,3,1), (6,6,2,1), (6,3,4,1), (6,3,2,2), (4,9,2,1), (4,6,3,1), (4,3,6,1), (4,3,3,2), (4,3,2,3). The extra factors that they give beyond the minimal (4,3,2,1) are, respectively, $2^4 5$, $2^2 3^3$, $2^2 5^2$, $2^2 7$, 3^6, $3^3 5$, 5^4, $5 \cdot 7$, 7^2.

36. 208. [4,0] Let $A_1 \leq A_2 \leq A_3$ denote the three sides. We must have $A_3 < A_1 + A_2$ and $A_3 = 100 - A_1 - A_2$. Thus $50 < A_1 + A_2$ and $A_2 \leq 100 - A_1 - A_2$. This yields

$$\max(A_1, 51 - A_1) \leq A_2 \leq \tfrac{100 - A_1}{2}$$

as the range of values for A_2. Then A_3 is uniquely determined by A_1 and A_2. As A_1 goes from 2 to 33, the number of admissible values of A_2 goes

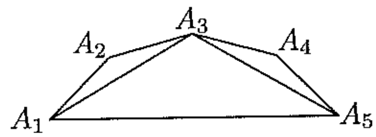
$$1, 1, 2, 2, 3, 3, \ldots, 12, 12, 12, 10, 9, 7, 6, 4, 3, 1.$$

The sum of these is 208. One way to quickly add them is to pair the opposite numbers among the first 24 and to pair opposite numbers among the last 12. This yields $16 \cdot 13$.

37. 8. [3,2] By looking at the graph of $f(x)$ for $0 \leq x < 1$, we see that the equation $f(x) = y$ has 1 solution, $x = 2y/3$, if $1/2 \leq y < 1$, and 2 solutions, $2y/3$ and $2(y+1)/3$, if $0 \leq y < 1/2$. First, $f(x) = 0$ implies $x = 0$ or $2/3$. Then, $f(f(x)) = 0$ implies $f(x) = 0$ or $2/3$, and so $x = 0$, $2/3$, or $4/9$. Next, $f(f(f(x))) = 0$ implies $f(x) = 0$, $2/3$, or $4/9$, and so $x = 0$, $2/3$, $4/9$, $8/27$, or $26/27$. Finally, $f(f(f(f(x)))) = 0$ has $2 + 1 + 2 + 2 + 1$ solutions.

38. $5 + 2\sqrt{3}$. [1,0] The interior angle at each A_i is 150° and so the area of each of the two skinny triangles in the diagram below is $\frac{1}{2} \cdot 2 \cdot 2 \sin 150 = 1$. If $s = A_1 A_3$, then

$s^2 = 2^2 + 2^2 - 2 \cdot 2 \cdot 2\cos 150 = 8 + 4\sqrt{3}$. Angle $A_1A_3A_5$ equals $150 - 15 - 15 = 120$. The area of triangle $A_1A_3A_5$ is $\frac{1}{2}s^2 \sin 120 = 3 + 2\sqrt{3}$. Thus the total area is $5 + 2\sqrt{3}$.



39. 5/8. [3,2] Let E denote the event that HTH occurs before THTH. If s denotes a sequence, let $P(E|\mathrm{s})$ denote the probability that E occurs given that the sequence starts with s. Let $x = P(E|H)$, i.e. the probability that E occurs given that the sequence starts with an H, and similarly $y = P(E|T)$. Then

$$x = \tfrac{1}{2}P(E|HH) + \tfrac{1}{4}P(E|HTT) + \tfrac{1}{4}P(E|HTH) = \tfrac{1}{2}x + \tfrac{1}{4}y + \tfrac{1}{4}$$

and

$$y = \tfrac{1}{2}P(E|TT) + \tfrac{1}{4}P(E|THH) + \tfrac{1}{8}P(E|THTT) + \tfrac{1}{8}P(E|THTH) = \tfrac{1}{2}y + \tfrac{1}{4}x + \tfrac{1}{8}y.$$

Here we have noted that whenever HH occurs, it is like starting over with an H, and similarly for TT. Solving these two equations yields $x = \frac{3}{4}$ and $y = \frac{1}{2}$. Thus the desired probability is $\frac{1}{2}x + \frac{1}{2}y = \frac{5}{8}$.

40. $\sqrt{5 + 2\sqrt{5}}$. [2,0] Recall that $s = (1 + \sqrt{5})/2$ and satisfies $s^2 = 1 + s$. Let (x, y) denote the coordinates of the distinguished point on the diagonal considered in Problem 26, if A is at $(0, 0)$. Then $x^2 + y^2 = s^2$ and $(x - s)^2 + y^2 = 1$. This implies that $s^2 - 2xs + s^2 = 1$, and so $x = (2s^2 - 1)/(2s) = (2s + 1)/(2s) = (1 + s)/2$. Then $y^2 = s^2 - (1 + s)^2/4 = (3s^2 - 2s - 1)/4 = (s + 2)/4$. The coordinates of the point C in the diagram above are $\frac{s+1}{s}(x, y) = (sx, sy)$. We have $sx = \frac{1}{2}(1 + s)s = s + \frac{1}{2}$. Thus the x-component of B is $1/2$. The y-component of B equals

$$\tfrac{s}{2}\sqrt{s + 2} = \tfrac{1}{2}\sqrt{(1 + s)(2 + s)} = \tfrac{1}{2}\sqrt{3 + 4s} = \tfrac{1}{2}\sqrt{5 + 2\sqrt{5}}.$$

The desired tangent now equals 2 times the y-component of B, which is as claimed.