# Special Topics in Relaxation in Glass and Polymers

# Lecture 3: Complex exponential function, Fourier and Laplace transforms

**Dr. Chris Cox**

***Department of Mathematical Sciences***

**Clemson University**

## Caveat

Details that we would cover in more depth if time permitted:

- Conditions guaranteeing existence and uniqueness of the transforms
- Conditions on arguments in specific cases

## Supplementary reading

- Erwin Kreysig, *Advanced Engineering Mathematics*, 5th edition.
    - Chapter 12:  Complex Numbers. Complex Analytic Functions
    - Chapter 5: Laplace Transformation
- Gilbert Strang, *Introduction to Applied Mathematics*
    - Section 4.3: Fourier Integrals
- George W. Scherer, *Relaxation in Glass and Composite*, 1992
    - Appendix A: Laplace Transform
- Many internet resources, e.g. *Wikipedia* articles on the transforms.  Use multiple sources.

# 3.1 Complex Exponential Function

**Outline:**

- **Complex numbers**
  - **Definitions**
  - **Arithmetic operations**
  - **Properties**
  - **Polar form**
- **Complex functions**
- **The exponential function**
- **Exercise set 1**

*The shortest route between two truths in the real domain passes through the complex domain.*

- Jacques Salomon Hadamard (1865-1963)

# Complex Numbers

**Representation of complex numbers:**

- **an *ordered pair* of real numbers:**

$$z = (x, y)$$

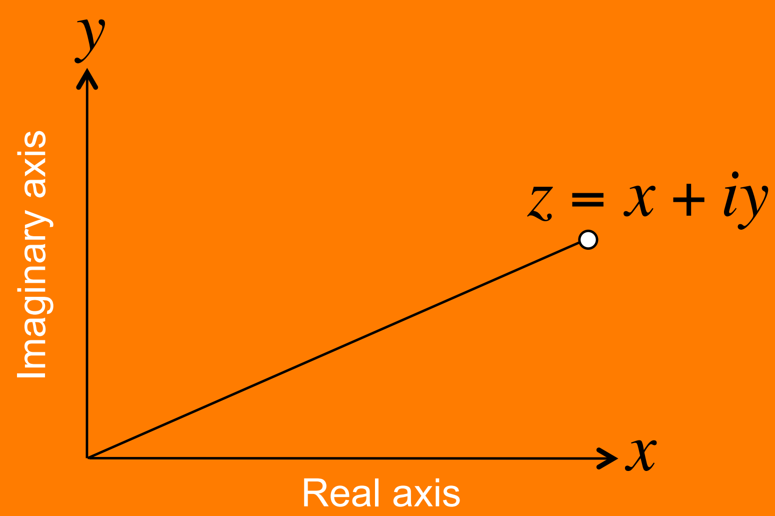
(consisting of *real part* and *imaginary part*)

- **using the *imaginary unit*:**

$$z = x + iy$$

**where** $i^2 = -1$

- **representation in the complex plane:**



# Complex Numbers

**Arithmetic operations (given** $z_1 = x_1 + iy_1$ **and** $z_2 = x_2 + iy_2$ **)**

- **addition and subtraction:**

$$z = z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

- **multiplication:**

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

- **division:**

$$z = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)} \frac{(x_2 - iy_2)}{(x_2 - iy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

**i.e. use the *complex conjugate* of the denominator** $\overline{x_2 + iy_2} = x_2 - iy_2$

# Complex Numbers

## Properties

- commutative and associative laws for addition and multiplication
- distributive law

same as for real numbers

## Polar form

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

- *modulus* or *absolute value*:

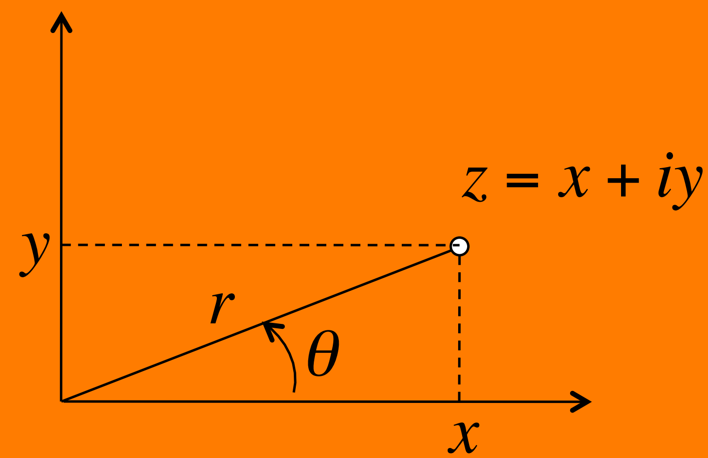
$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

- *argument*:

$$\theta = \arg(z)$$

- *principle value* of the argument:

$$-\pi < \theta \le \pi$$



## Complex Functions

• A *function* $f$ defined on a set $S$ of complex numbers assigns to each $z$ in $S$ a unique complex number $w$, and we write

$$w = f(z) = u(x,y) + iv(x,y)$$

where $u(x,y)$ and $v(x,y)$ are real functions, and are also the real and imaginary parts of $z$.

• Continuity, differentiability, and derivative of $f(z)$ are defined in an analogous manner to the properties of real functions.

• The rules of real differential calculus carry over to complex functions, e.g.

$$\frac{d}{dz}(z^3) = 3z^2$$

• The function $f(z)$ is *analytic in a domain D* if $f(z)$ is defined and differentiable at all points in z, and $f(z)$ is *analytic at a point* $z_0$ *in D* if $f(z)$ is analytic in a neighborhood of $z_0$.

# Complex Functions

**Theorem (Cauchy-Riemann equations)**

**If function $f(z) = u(x,y) + iv(x,y)$ is defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at $z$, then the first order partial derivatives of the components of $f(z)$ exist and satisfy the *Cauchy-Riemann* equations:**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

**As a consequence, wherever the partial derivatives do not exist or the C-R equations are not satisfied, then $f(z)$ is not analytic.**

# Complex Exponential Function

**Definition: For the complex number $z = x + iy$, the *exponential function* is defined as**

$$e^{z} = e^{x}(\cos y + i\sin y) \qquad \textbf{(1)}$$

**Remarks:**

- **If $x = 0$, then we have *Euler's formula*, which says that for a real number $y$,**

$$e^{iy} = \cos y + i\sin y$$

- **Some sources begin with Euler's formula, and derive (1) using the formula**

$$e^{x+iy} = e^{x}e^{iy}$$

**Properties:**

- **The polar form can now be written**

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

- **Periodicity:** $re^{i\theta} = re^{i(\theta+2k\pi)}$ **for any integer $k$**

  **i.e.** $z = re^{i\theta}$ **is periodic with the *imaginary period* $2\pi i$**

# Complex Exponential Function

## More Properties

- **Multiplication and Division: For complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$**

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \qquad z_2 \neq 0$$

- **Powers:**

$$z^n = (x + iy)^n = r^n e^{in\theta}$$

- **De Moivre's formula**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

- **For the modulus**

$$\left|e^z\right| = \left|e^{x+iy}\right| = e^x \qquad \left|e^{iy}\right| = 1$$

- **Derivative**

$$\frac{d}{dz}(e^z) = e^z \qquad \frac{d}{dt}(e^{(a+bi)t}) = (a+bi)e^{(a+bi)t}$$

# Exercise Set 1

a) For $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$, find $z_1 / z_2$.

b) For $z = 2 + 5\pi i$, find the value of $e^z$.

c) True or False? The function $f(z) = \bar{z} = x - iy$ is nowhere analytic.

Hint: Cauchy-Riemann

# 3.2 The Fourier Transform

**Outline:**

- Motivation/background
- Definition
- Some common transforms
- Properties/rules
- Derivatives, integrals, and shifts
- Exercise set 2

**Jean Baptiste Joseph Fourier**

**Born: 21 March 1768 in Auxerre, Bourgogne, France**
**Died: 16 May 1830 in Paris, France**

# 3.2 The Fourier Transform

**Motivation – applications**

- **Solving linear differential equations and partial differential equations by translating them into *algebraic equations*, for example**

- **Electrical engineering – analysis of voltage and currents**

- **Digital signal and image processing**

- **Origin of the concept: *Théorie analytique de la chaleur* (Analytical Theory of Heat), which Fourier published in 1822.**

  **(The paper Fourier submitted to the French Academy of Science in 1807 on the problem of heat conduction was rejected for its lack of rigor, according to Ahmed I. Zayed, in *Handbook of function and generalized function transformations*)**

# The Fourier Transform

## Definition

- Let the function $f$ be integrable on the real line, i.e. $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$

Then the *Fourier transform* of $f$ is a function $\hat{f}(k)$ (depending on *angular frequency* $k$ ) defined as the improper integral

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

- inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk$$

- Other forms

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk \qquad \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

(symmetric forms)

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx}\, dx \qquad f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{2\pi ikx}\, dk$$

(using *oscillation frequency* $k$ )

- Other common notations $F(k)$, $F(\xi)$, $\Phi_{\xi}(f(x))$

C. L. Cox
Relaxation Processes in Glass
and Polymers, Lecture 3

# The Fourier Transform

## Some common transforms

| | $f(x)$ | $\hat{f}(k)$ |
|---|---|---|
| | $\delta(x)$ | $\int_{-\infty}^{\infty}\delta(x)e^{-ikx}dx = 1$ |
| | $1$ | $2\pi\delta(k)$ |
| | $\cos(ax)$ | $\pi(\delta(k-a)+\delta(k+a))$ |
| | $\sin(ax)$ | $i\pi(\delta(k+a)-\delta(k-a))$ |
| **sign function** | $sgn(x) = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$ | $\dfrac{2}{ik}$ |
| | $\dfrac{1}{x}$ | $-i\pi sgn(k)$ |
| **step function** | $u(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$ | $\pi\delta(k) + \dfrac{1}{ik}$ |

**The *dirac delta function*, $\delta(x)$, for our purposes, is defined by its effect when it appears in a product in an integrand:** $\int_{-\infty}^{\infty}\delta(x-a)f(x)dx = f(a)$

# The Fourier Transform

C. L. Cox
Relaxation Processes in Glass
and Polymers, Lecture 3

## Properties/rules

| | $f(x)$ | $\hat{f}(k)$ |
|---|---|---|
| • **Linearity** | $ag(x) + bh(x)$ | $a\hat{g}(k) + b\hat{h}(k)$ |
| • **Convolution** | $(g * h)(x) = \int_{-\infty}^{\infty} g(x-y)h(y)dy$ | $\hat{g}(k)\hat{h}(k)$ |
| • **Transform** | $\hat{g}(x)$ | $2\pi g(-k)$ |
| • **Product** | $g(x)h(x)$ | $\frac{1}{2\pi}(\hat{g} * \hat{h})(k)$ |
| | **real even function** | **real even function** |
| | **real odd function** | **imaginary odd function** |

**Parseval's theorem:** $$2\pi \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k)dk$$

**Plancheral theorem:** $$2\pi \int_{-\infty}^{\infty} \left|f(x)\right|^2 dx = \int_{-\infty}^{\infty} \left|\hat{f}(k)\right|^2 dk$$

# The Fourier Transform

## Derivatives, integrals, and shifts

| $f(x)$ | $\hat{f}(k)$ |
|---|---|
| $\dfrac{d^n g(x)}{dx^n}$ | $(ik)^n \hat{g}(k)$ |
| $x^n g(x)$ | $i^n \dfrac{d^n \hat{g}(k)}{dk^n}$ |
| $\int_a^x g(t)\,dt$ | $\dfrac{1}{ik}\hat{g}(k) + c\delta(k)$ |
| $g(x-a)$ | $e^{-iak}\hat{g}(k)$ |
| $e^{ixa} g(x)$ | $\hat{g}(k-a)$ |

# Exercise Set 2

a) Find the Fourier transform of $f(x) = e^{2xi}\delta(x)$

b) Find the inverse Fourier transform of $F(k) = \dfrac{2}{i(k-3)}$

c) Find the Fourier transform of $f(x) = \int_{-\infty}^{\infty} \cos(x-y)\sin(y)\,dy$

# 3.3 The Laplace Transform

**Outline:**

- Background
- Definition
- Some common transforms
- Properties/rules
- Derivatives, integrals, and shifts
- Exercise set 3


**Pierre-Simon Laplace (1749-1827)**

# The Laplace Transform

## Applications

- Solving differential equations (ODE‘s and PDE‘s) by converting them to algebraic equations
  - analysis of dynamical systems
  - electrical circuits

- Evaluating integrals of certain forms

- *The Laplace transform sees a lot of use in probability theory, where Laplace first found it.*

- Paul J. Nahin, Behind the Laplace transform,
*IEEE Spectrum, March 1991.*

# The Laplace Transform

## Definitions

- For a function *f(t)* which is defined for all $t \geq 0$, the (unilateral) ***Laplace transform*** is defined as the improper integral

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

for those (possibly complex) values of $s$ for which the integral makes sense.

- Inverse Laplace transform – not as straightforward to define as for the Fourier transform, because it is an integral over the complex plane. For completeness, we provide a definition, though the formula is not widely used directly. The ***inverse Laplace transform of*** $F(s)$ is defined as

$$f(t) = \int_{c-i\infty}^{c+i\infty} e^{st} F(s)\,ds$$

with conditions placed on the real number $c$.

- Other common notations: $F(s) = \mathcal{L}\{f(t)\} = f^*(s)$ $\qquad f(t) = \mathcal{L}^{-1}\{F(s)\}$

# The Laplace Transform

## Some common transforms

| $f(t)$ | $F(s)$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $t$ | $\frac{1}{s^2}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $\delta(t - a)$ | $e^{-as}$ |
| $\cos(\omega t)$ | $\frac{s}{s^2 + \omega^2}$ |
| $\sin(\omega t)$ | $\frac{\omega}{s^2 + \omega^2}$ |

# The Laplace Transform

**Properties/rules**

| | $\underline{f(t)}$ | $\underline{F(s)}$ |
|---|---|---|
| **linearity** | $ag(t)+bh(t)$ | $aG(s)+bH(s)$ |
| **convolution** (note limits on integral) | $(g*h)(t)=\int_0^t g(t-y)h(y)dy$ | $G(s)H(s)$ |
| **time scaling** | $g(at),\ a>0$ | $(1/a)G(s/a)$ |
| **continuous periodic function** | $g(t)=g(t+p)$ | $\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)dt, \quad s>0$ |

# The Laplace Transform

## Derivatives, integrals, and shifts

| | $f(t)$ | $F(s)$ |
|---|---|---|
| (prove using integration by parts) | $g'(t) = \dfrac{dg}{dt}$ | $sG(s) - g(0)$ |
| | $g^{(n)}(t) = \dfrac{d^n g}{dt^n}$ | $s^n G(s) - s^{n-1} g(0) - s^{n-2} g'(0) - \cdots - g^{(n-1)}(0)$ |
| | $tg(t)$ | $-G'(s)$ |
| | $\dfrac{g(t)}{t}$ | $\int_s^\infty F(v)dv$ |
| | $\int_0^t g(y)dy$ | $\dfrac{1}{s} G(s)$ |
| | $e^{at} g(t)$ | $G(s-a)$ |
| | $g(t-a)u_a(t) = \begin{cases} 0 & t < a \\ g(t-a) & t > a \end{cases}$ | $e^{-as} G(s)$ |

# Exercise Set 3

a) **Find the Laplace transform of** $f(t) = 3t + 4$

b) **Find the inverse Laplace transform of** $\dfrac{1}{s^3 + 4s}$

c) **Find the inverse Laplace transform of** $\dfrac{4(1 - e^{-3s})}{s}$